Canadian Mathematical Bulletin

canadien de mathématiques

Tome 39 numéro 4, décembre 1996

CANADIAN MATHEMATICAL BULLETIN
BULLETIN CANADIEN DE MATHÉMATIQUES

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IMPRIMÉ AU CANADA PAR LES PRESSES DE L'UNIVERSITÉ DE TORONTO

CN ISSN 0008-4395

suite en troisième couverture.

Canad. Math. Bull. Vol. **39** (4), 1996 pp. 385–389

ON THE REDUCED PRODUCT CONSTRUCTION

Dedicated to the memory of Peter Fantham

PETER FANTHAM, IOAN JAMES AND MICHAEL MATHER

ABSTRACT. Under certain conditions the reduced product space JX of a space X has the same homotopy type as $\Omega\Sigma X$, the loop-space on the suspension of X. Several proofs can be found in the literature. The original proof [6] made unnecessarily strong assumptions. Later, in the last chapter of [3], tom Dieck, Kamps and Puppe gave a proof under much weaker conditions and showed that they could not be further weakened. The question then arose as to whether the reduced product construction could be generalized to provide combinatorial models not only of $\Omega\Sigma X$ but also of $\Omega^2\Sigma^2 X$, $\Omega^3\Sigma^3 X$ and so on. This was answered in the affirmative by May [10], using ideas of Boardman and Vogt [1], and the construction was further developed by Segal[1] [11] and others.

The present note, however, is not concerned with these generalizations. Its purpose is to give a simple proof of the original result, without striving for maximum generality, and to show that the same method can be used to prove an equivariant version of the reduced product theorem and hence a fibrewise version. Fibrewise versions of the reduced product theorem have previously been given by Eggar [5] and, more recently, by one of us [8] but it may be useful to have a relatively simple treatment which is adequate for the majority of applications.

To begin with we work in the category of pointed spaces. Unless otherwise stated maps and homotopies are basepoint-preserving. Suspension always means reduced suspension. Later we shall turn to the equivariant theory, but here again the action is to be basepoint-preserving.

Let X be a space with basepoint x_0, which we assume to be closed. As a set, the reduced product space JX may be described as the free monoid on X, with x_0 acting as neutral element. Thus a point of JX may be represented by a finite sequence of points of X. Sequences with not more than n terms form a subset $J^nX \subset JX$ $(n = 0, 1, \ldots)$. We topologize J^nX as a quotient space of the n-fold topological product $\prod^n X$. Then we obtain a sequence of spaces

$$J^0X \subset J^1X \subset \cdots \subset J^nX \subset \cdots,$$

where each member of the sequence is contained in the next as a closed subspace. Finally we topologize JX itself as the colimit[2] of the sequence; this does not mean (see (17.10)

This paper arose from unpublished work of the first and last authors dating from around 1972.

Received by the editors July 13, 1994.

AMS subject classification: 55P15, 55P35, 55P91.

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[1] Some of the results of [11], including those which are relevant here, need an additional numerability condition.

[2] A convenient reference for results about colimits is Vogt [12].

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of [3]) that the multiplication on JX is continuous. We prove

THEOREM 1. *Let X be a well-pointed compact Hausdorff space. Suppose that X can be covered by open sets each of which is contractible*[3] *in X to the basepoint. Then JX has the same homotopy type as $\Omega\Sigma X$.*

In the proof which follows we disregard basepoints. We shall show that JX and $\Omega\Sigma X$ have the same homotopy type in the non-pointed sense. For reasons given in (17.3) of [3] this will imply that they have the same homotopy type in the pointed sense.

First observe that the natural projection from $\prod^n X$ to $J^n X$ ($n = 0, 1, \ldots$) is proper, since X is compact, and hence the function

$$X \times J^n X \longrightarrow J^{n+1} X$$

defined by the multiplication on JX is continuous. Furthermore $X \times JX$ is the colimit of the sequence

$$X \times J^0 X \subset X \times J^1 X \subset \cdots \subset X \times J^n X \subset \cdots,$$

and so the function

$$T: X \times JX \longrightarrow JX$$

thus defined is continuous.

The first step in the proof of the theorem is to show that, for each n, the homotopy push-out of the cotriad

$$J^{n+1} X \longleftarrow X \times J^n X \xrightarrow{\pi_2} J^n X$$

is contractible, where the left-hand arrow is given by the restriction of T. This is trivial when $n = 0$; make the inductive hypothesis that it is true for some $n \geq 0$.
Write

$$X \triangleright J^n X = x_0 \times J^n X \cup X \times J^{n-1} X \subset X \times J^n X$$

and consider the diagram shown below.

$$\begin{tikzcd}
X \times J^n X \arrow[rr, "\pi_2"] \arrow[d] & & J^n X \arrow[d] \\
X \triangleright J^{n+1} X \arrow[r] \arrow[d] & x_0 \times J^{n+1} X \arrow[r] \arrow[d] & J^{n+1} X \arrow[d] \\
J^{n+1} X \arrow[r] & J^{n+2} X \arrow[r] & x_0
\end{tikzcd}$$

The outer square is a homotopy push-out, by the inductive hypothesis. Also the top half of the diagram is a push-out in the topological sense, hence a homotopy push-out (since the inclusion $J^n X \longrightarrow J^{n+1} X$ is a cofibration so is the inclusion $X \times J^n X \longrightarrow X \triangleright J^{n+1} X$). Similarly the bottom left-hand square is a homotopy push-out. Hence it follows from

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[3] Here and in Theorem 2 contractions are not required to preserve basepoints.

Theorem 10(ii) of Mather [9] that the bottom right-hand square is also a homotopy push-out, which proves the inductive step.

Now the colimit of this sequence of contractible homotopy push-outs is just the homotopy push-out of the cotriad

$$JX \xleftarrow{T} X \times JX \xrightarrow{\pi_2} JX.$$

We conclude, therefore, that this space is contractible.[4] So far we have only used the assumptions that X is well-pointed and compact Hausdorff.

Now consider the diagram shown below, which depicts two adjacent faces of a cube.

$$\begin{tikzcd}
X \times JX \arrow[rr, "\pi_2"] \arrow[dr, "T"] \arrow[dd] & & JX \arrow[dd] \\
& JX \arrow[dd] & \\
X \arrow[rr] \arrow[dr] & & x_0 \\
& x_0 &
\end{tikzcd}$$

Obviously the back face in the diagram is a homotopy pull-back. We assert that the other face is also a homotopy pull-back. This amounts to showing that the "shearing map"

$$\xi : X \times JX \to X \times JX,$$

with components (π_1, T), is a fibrewise homotopy equivalence, where $X \times JX$ is regarded as a fibre space over X through the first projection. However X is path-connected and of finite category, and so this follows from Dold's theorem (6.3) of [4].

Thus both faces are homotopy pull-backs and so, completing the diagram of the cube as shown below, we can apply Theorem 11 of Mather [9] and conclude that the front face is a homotopy pull-back, from which Theorem 1 follows at once.

$$\begin{tikzcd}
X \times JX \arrow[rr] \arrow[dr] \arrow[dd] & & JX \arrow[dr] \arrow[dd] & \\
& JX \arrow[rr] \arrow[dd] & & x_0 \arrow[dd] \\
X \arrow[rr] \arrow[dr] & & x_0 \arrow[dr] & \\
& x_0 \arrow[rr] & & \Sigma X
\end{tikzcd}$$

We turn now to the equivariant version of the theorem, which does not seem to have been treated in the literature. Specifically let X be a (pointed) G-space, where G is a compact group. Then JX and $\Omega\Sigma X$ are also G-spaces, and we state

THEOREM 2. *Let X be an equivariantly well-pointed compact Hausdorff G-space, where G is a compact Lie group. Suppose that X can be covered by invariant open sets each of which is G-contractible to the basepoint. Then JX has the same G-homotopy type as $\Omega\Sigma X$.*

The proof of Theorem 2 proceeds on the same lines as the proof of Theorem 1. An equivariant version of Mather's theory is needed, but this is completely routine. An equivariant version of Dold's theorem is also needed, but this poses no problems.[5] It seems unnecessary, therefore, to go through the details.

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[4] Here and in Theorem 2 contractions are not required to preserve basepoints.

The supposition is equivalent to the assumption that the G-category of X is defined, in the pointed sense, and is therefore finite, by compactness. This is true, for example, if X is a finite G-complex such that the fixed point set X^H is connected for all closed subgroups H of G, as is the case when X is a double suspension. Thus take G to be the orthogonal group $O(n-1)$ acting on the sphere S^n in the usual way, where $n \geq 1$, so that the poles are left fixed and a point x_0 on the equator is also left fixed. Cover S^n by the enlarged hemispheres, which are $O(n-1)$ contractible to their respective poles, and then deform the poles into x_0, which we take as basepoint, along the line of longitude. We see that the condition is fulfilled in this case.

Finally we turn to the fibrewise version of the reduced product theorem. We work in the category of sectioned fibrewise spaces over a given base space B and require the standard section to be closed. Then the fibrewise reduced product space J_BX is defined as in [8] for each fibrewise space X over B. The fibres of J_BX are just the reduced product spaces of the corresponding fibres of X. We prove

THEOREM 3. *Let X be a (sectioned) fibre bundle over the base space B with compact structure Lie group G and fibre the (pointed) G-space A. Suppose that A satisfies the conditions of Theorem 2. Then J_BX has the same fibrewise homotopy type as $\Omega_B\Sigma_BX$.*

Let P be the principal G-bundle over B associated with X. Then (see [7]) P determines a functor $P_\#$ which transforms each G-space into the associated bundle with that fibre, and similarly with G-maps and G-homotopies. Moreover, $P_\#$ transforms the equivariant reduced product space JA into the fibrewise reduced product space J_BX, and similarly with the suspension and loop-space. Since JA has the same equivariant homotopy type as $\Omega\Sigma A$, by Theorem 2, it follows at once that J_BX has the same fibrewise homotopy type as $\Omega_B\Sigma_BX$.

The original version of the fibrewise reduced product theorem is due to Eggar [5] but conditions are imposed which are rather inconvenient in practice. More recently one of us [8] has published a fibrewise version of the proof given by tom Dieck, Kamps and Puppe [3] in the ordinary case. This does not assume local triviality. However, Theorem 3 seems adequate for most applications and its proof is a good deal simpler than the alternatives. For example it applies in the case of an orthogonal sphere-bundle which admits a pair of mutually orthogonal sections.

DEDICATION. This paper is dedicated to the memory of Peter Fantham, who died on June 10th, 1992. Peter was an editor of the Canadian Journal of Mathematics from 1971 to 1976.

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[5] See, for example, the last exercise in (7.5) of tom Dieck [2].

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Canad. Math. Bull. Vol. **39** (4), 1996 pp. 390–401

A FULL DESCRIPTIVE DEFINITION OF THE GAGE INTEGRAL

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ABSTRACT. We consider a specific Riemann type integral, called the *gage integral*. Using variational measures, we characterize all additive functions of intervals that are indefinite gage integrals. The characterization generalizes the descriptive definition of the classical Denjoy-Perron integral to all dimensions.

The gage integral, introduced in [6, Chapter 11], is a Riemann type integral which extends, in a natural way, the most important properties of the classical Denjoy-Perron integral to higher dimensions. In particular, it integrates the partial derivatives of differentiable functions so that the usual Gauss-Green formula is satisfied. Our main result (Theorem 4.4) gives a complete characterization of additive functions of intervals that are indefinite gage integrals. The well-known characterization in dimension one ([7, Chapter VIII]) depends on the order structure of the real line, and does not permit a direct extension to a multidimensional situation.

We employ suitably modified techniques of variational measures (see [8]), and rely heavily on the Ward theorem (Theorem 3.1). Thus any generalization of our results to Riemann type integrals defined by means of figures or sets of finite perimeter may require an appropriate generalization of Ward's theorem.

The paper is organized as follows. After some necessary preliminaries in Section 1, we give a detailed motivation for the gage integral in Section 2. The precise definition of the gage integral is given; for its properties, however, we refer to [6]. In addition, the reader is assumed to have some familiarity with the one-dimensional Henstock-Kurzweil integral. Section 3 is devoted to variational measures, whose application to the gage integral in Section 4 yields the desired descriptive definition. As a corollary, we obtain a condition under which the variational measure associated to the flux of a continuous vector field is absolutely continuous and σ-finite.

1. **Preliminaries.** The set of all real numbers is denoted by $\mathbf{R}$, and the ambient space of this paper is $\mathbf{R}^m$ where m is a positive integer. The metric in $\mathbf{R}^m$ is induced by the maximum norm. For a set $E \subset \mathbf{R}^m$, we denote by ∂E, $d(E)$, and $|E|$ the boundary, diameter, and Lebesgue measure of E, respectively. The words "measure" and "measurable" as well as the expressions "almost all" and "almost everywhere" always refer to the Lebesgue

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The first author was supported in part by a grant from the MURST of Italy.
The second author was supported in part by a grant from the CNR of Italy.
The third author was supported in part by a grant from the NSERC of Canada.
Received by the editors October 28, 1994.
AMS subject classification: Primary: 26A39, 26B30.
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measure in $\mathbf{R}^m$. As usual, a set $E \subset \mathbf{R}^m$ is called *negligible* whenever $|E| = 0$. We say subsets A and B of $\mathbf{R}^m$ *overlap* if their intersection is not negligible.

The $(m-1)$-dimensional Hausdorff measure in $\mathbf{R}^m$ is denoted by $\mathcal{H}$, and a set $T \subset \mathbf{R}^m$ of σ-finite measure $\mathcal{H}$ is called *thin*. We note that by definition the 0-dimensional Hausdorff measure is the counting measure, and hence thin subsets of $\mathbf{R}$ coincide with countable sets.

Unless stated otherwise, a function is always real-valued. A nonnegative function δ on a set $E \subset \mathbf{R}^m$ is called a *gage* or an *essential gage* (abbreviated as *e-gage*) on E whenever its *null set* $N_\delta = \{x \in E : \delta(x) = 0\}$ is thin or negligible, respectively.

A *cell* is a compact nondegenerate interval, and a *figure* is a finite, possibly empty, union of cells. If $A \subset \mathbf{R}^m$ is a figure, the numbers

$$\|A\| = \mathcal{H}(\partial A) \quad \text{and} \quad s(A) = \frac{|A|}{[d(A)]^m}$$

are called the *perimeter* and *shape* of A, respectively.

A *partition* is a collection $P = \{(A_1, x_1), \dots, (A_p, x_p)\}$ where $A_1, \dots, A_p$ are nonoverlapping cells, and $x_i \in A_i$ for $i = 1, \dots, p$. Given $E \subset \mathbf{R}^m$, we say P is

- a partition *in* E if $\bigcup_{i=1}^p A_i \subset E$,
- a partition *of* E if $\bigcup_{i=1}^p A_i = E$,
- *anchored* in E if $\{x_1, \dots, x_p\} \subset E$.

If $s(A_i) > \varepsilon$ for an $\varepsilon > 0$ and $i = 1, \dots, p$, we say that P is *ε-shapely*; when $\varepsilon < 1$, every partition in dimension one is ε-shapely. For a nonnegative function δ on E, a partition P anchored in E is called *δ-fine* whenever $d(A_i) < \delta(x_i)$ for $i = 1, \dots, p$.

2. **The gage integral.** To motivate the definition of the gage integral, we begin with a modified definition of the classical Henstock-Kurzweil integral (see [6, Section 6.1]), referred to as the HK-integral. Recall that a function F on a cell $A \subset \mathbf{R}$ defines an additive function on the family of all subcells of A by setting $F([c,d]) = F(d) - F(c)$ for each cell $[c,d] \subset A$. Moreover, this function of cells has a unique extension, still denoted by F, to an additive function on the family of all subfigures of A.

DEFINITION 2.1. A function f is HK-*integrable* on a cell $A \subset \mathbf{R}$ if there is a function F on A satisfying the following condition: given $\varepsilon > 0$, there is a positive function δ on A such that

$$\sum_{i=1}^{p} \left| f(x_i)|A_i| - F(A_i) \right| < \varepsilon$$

for each δ-fine partition $\{(A_1, x_1), \dots, (A_p, x_p)\}$ in A.

The equivalence of Definition 2.1 and the original definition of Henstock and Kurzweil ([6, Definition 6.1.1]) follows immediately from Henstock's lemma ([6, Lemma 2.3.1 and Proposition 6.1.5]). It is also clear that F, viewed as an additive function of cells, is the *indefinite* HK-*integral* of f. Moreover, a moment's reflection reveals that F, as a function of points, is continuous on A. Using this it is easy to prove the following proposition.

Proposition 2.2. *A function f is* HK*-integrable on a cell $A \subset \mathbf{R}$ if and only if there is a continuous function F on A satisfying the following condition: given $\varepsilon > 0$, there is a gage δ on A such that*

$$\sum_{i=1}^{p} \left| f(x_i)|A_i| - F(A_i) \right| < \varepsilon$$

for each δ-fine partition $\{(A_1, x_1), \dots, (A_p, x_p)\}$ in A. The function F, viewed as an additive function of cells, is the indefinite integral of f.

Proof. As the converse is obvious, assume the condition of the proposition is satisfied. Choose an $\varepsilon > 0$ and find a corresponding gage δ on A with null set $N_\delta = \{z_1, z_2, \dots\}$. Define a positive function Δ on A so that $\Delta(x) = \delta(x)$ for each $x \in A - N_\delta$ and

$$\left| f(z_n)|B| - F(B) \right| < \varepsilon 2^{-n}$$

for $n = 1, 2, \dots$ and each cell $B \subset A$ with $z_n \in B$ and $d(B) < \Delta(z_n)$. Such a choice of Δ is possible since F is a continuous function. Now for a Δ-fine partition $\{(A_1, x_1), \dots, (A_p, x_p)\}$ in A, we obtain

$$\begin{aligned}\sum_{i=1}^{p} \left| f(x_i)|A_i| - F(A_i) \right| &= \sum_{x_i \notin N_\delta} \left| f(x_i)|A_i| - F(A_i) \right| + \sum_{n=1}^{\infty} \sum_{x_i = z_n} \left| f(x_i)|A_i| - F(A_i) \right| \\ &\le \varepsilon + 2\varepsilon \sum_{n=1}^{\infty} 2^{-n} = 3\varepsilon,\end{aligned}$$

and the proposition is proved.

A direct generalization of the HK-integral to higher dimensions is straightforward but disappointing: the most attractive feature of the one-dimensional HK-integral, its ability to integrate derivatives, is lost in the process. It was first recognized by Mawhin (see [5]) that in order to integrate partial derivatives of differentiable functions, we must employ ε-shapely partitions. However, doing this simplistically leads to an integral without the usual additive properties. Since Mawhin's work several authors tried to reinstate the additivity by inventing highly technical, often artificial, variations of the multidimensional HK-integral. In this section, we show that Proposition 2.2 leads to a very natural higher dimensional generalization of the HK-integral called the *gage integral*. The gage integral has good additive properties and still integrates partial derivatives.

The only concept in Proposition 2.2 whose generalization to higher dimensions is not obvious is that of the continuous function F. Indeed, a continuous function on a one-dimensional cell must be replaced by a *continuous additive function* on the family of all subfigures of a multidimensional cell. The elementary observation below paves the way.

Observation 2.3. A function F on a cell $A \subset \mathbf{R}$ is continuous if and only if the associated function of figures has the following property: given $\varepsilon > 0$, there is an $\eta > 0$ such that $|F(B)| < \varepsilon$ for each figure $B \subset A$ with $\|B\| < 1/\varepsilon$ and $|B| < \eta$.

The condition stated in Observation 2.3 remains meaningful for additive functions defined on subfigures of a cell $A \subset \mathbf{R}^m$, and thus suggests our next definition.

DEFINITION 2.4. An additive function F on the family of all subfigures of a cell $A \subset \mathbf{R}^m$ is called *continuous* whenever given $\varepsilon > 0$, there is an $\eta > 0$ such that $|F(B)| < \varepsilon$ for each figure $B \subset A$ with $\|B\| < 1/\varepsilon$ and $|B| < \eta$.

To further motivate Definition 2.4, we describe the smallest topology τ on the family $\mathcal{A}$ of all subfigures of a cell $A \subset \mathbf{R}^m$ such that each continuous additive function on $\mathcal{A}$ is τ-continuous. There is a metric ρ on $\mathcal{A}$ defined by

$$\rho(B, C) = |(B - C) \cup (C - B)|$$

for all figures $B, C \subset A$, and an additive function F on $\mathcal{A}$ is ρ-continuous if and only if it is absolutely continuous. For $n = 1, 2, \ldots$, we let $\mathcal{A}_n = \{B \in \mathcal{A} : \|B\| \leq n\}$, and define τ as the largest topology on $\mathcal{A}$ for which all imbeddings $(\mathcal{A}_n, \rho) \hookrightarrow (\mathcal{A}, \tau)$ are continuous. It is easy to verify that τ is a sequential topology induced by a nonmetrizable uniformity. With some effort one can show that the completion of the space $(\mathcal{A}, \tau)$ consists of all BV subsets of A (see [3, Chapter 5]).

DEFINITION 2.5. A function f is *gage integrable* on a cell $A \subset \mathbf{R}^m$ if there is a continuous additive function F defined on the family of of all subfigures of A satisfying the following condition: given $\varepsilon > 0$, there is a gage δ on A such that

$$\sum_{i=1}^{p} \Big| f(x_i)|A_i| - F(A_i) \Big| < \varepsilon$$

for each ε-shapely δ-fine partition $\{(A_1, x_1), \ldots, (A_p, x_p)\}$ in A.

According to [6, Theorem 11.4.9], the gage integrability (abbreviated as g-integrability) of the above definition coincides with that defined in [6, Definition 11.4.1]. In particular, the function F is uniquely determined by f, and we call it the *indefinite g-integral* of f in A.

Every Lebesgue integrable function on a cell is g-integrable and the two indefinite integrals coincide ([6, Theorem 11.4.5]). It follows that if two functions on a cell are equal almost everywhere, then the g-integrability of one implies that of the other, and their integrals are equal ([6, Corollary 11.4.7]). Using this fact, we can and will, in the obvious way, introduce g-integrability and the g-integral for extended real-valued functions defined almost everywhere in a cell.

For additional properties of the g-integral, including the additivity and g-integrability of partial derivatives, we refer to [6, Chapter 11] (see also [6, Theorem 12.8.5]).

3. **Critical variation.** Let F be an additive function on the family of all subcells of a cell $A \subset \mathbf{R}^m$, and let $x \in A$. The *lower derivate* of F at x is the extended real number

$$\underline{F}(x) = \inf_{\alpha > 0} \sup_{\beta > 0} \left[\inf \frac{F(B)}{|B|} \right]$$

where the infimum in the brackets is taken over all cells $B \subset A$ with $x \in B$, $d(B) < \beta$, and $s(B) > \alpha$. The extended real-valued function $x \longmapsto \underline{F}(x)$ on A is denoted by $\underline{F}$. When $\underline{F}(x) = -\underline{(-F)}(x)$ is a real number, we say that F is *derivable* at x, and call this common value the *derivate* of F at x, denoted by $F'(x)$. By F' we denote the function $x \longmapsto F'(x)$ defined on the set of all $x \in A$ at which F is derivable. Note that our concept of derivation coincides with the *ordinary derivation* introduced in [7, Chapter IV, Section 2],

The pivotal result for our exposition is the theorem of Ward, whose proof can be found in [7, Chapter IV, Section 11].

THEOREM 3.1 (WARD). *Let F be an additive function on the family of all subcells of a cell $A \subset \mathbf{R}^m$, and let $E = \{x \in A : \underline{F}(x) > -\infty\}$. Then F is derivable at almost all $x \in E$.*

Let F be again an additive function on the family of all subcells of a cell $A \subset \mathbf{R}^m$. Given a set $E \subset A$, $\alpha > 0$ and a nonnegative function δ on E, let

$$F^{\alpha,\delta}(E) = \sup \sum_{i=1}^{p} |F(A_i)|$$

where the supremum is taken over all partitions $\{(A_1, x_1), \ldots, (A_p, x_p)\}$ in A anchored in E that are δ-fine and α-shapely. The *critical variation* or the *essential critical variation* of F on E is the extended real number

$$\sup_{\alpha>0} \inf_{\delta} F^{\alpha,\delta}(E)$$

where the infimum is taken over all gages or essential gages δ on E, respectively; it is denoted by $F^{*}(E)$ or $F^{e*}(E)$, respectively. An easy verification reveals that the extended real-valued functions

$$F^{*}: E \longmapsto F^{*}(E) \quad \text{and} \quad F^{e*}: E \longmapsto F^{e*}(E)$$

are metric measures in A (see [4, Chapter 1]), and that the measure F^{e*} is absolutely continuous with respect to the Lebesgue measure. As each gage is an e-gage, we have $F^{e*} \leq F^{*}$.

DEFINITION 3.2. An additive function F on the family of all subcells of a cell $A \subset \mathbf{R}^m$ is called BV_{*} or BV_{e*} whenever A is of σ-finite measure F^{*} or F^{e*}, respectively.

In dimension one the essential critical variation was studied in [2] and [1]. The one-dimensional version of the following theorem was proved in [1] by essentially the same techniques we use below.

THEOREM 3.3. *Let F be an additive function on the family of all subcells of a cell $A \subset \mathbf{R}^m$. Then, for any measurable set $E \subset A$, we have*

$$F^{e*}(E) = \int_E |\underline{F}(x)|\, dx$$

where the integral is the usual Lebesgue integral in $\mathbf{R}^m$.

PROOF. The Lebesgue integral $\int_A |\underline{F}(x)|\,dx$ exists (possibly equal to $+\infty$), since by [7, Chapter IV, Theorem 4.2], the extended real-valued function $\underline{F}$ is measurable on A.

Assume first that $\int_E |\underline{F}(x)|\,dx < F^{e*}(E)$ and find an $\alpha > 0$ so that

$$\int_E |\underline{F}(x)|\,dx < F^{\alpha,\delta}(E) \tag{1}$$

for every e-gage δ on E. According to our assumption, $\underline{F}$ is finite almost everywhere in E, and so by Ward's theorem, F is derivable almost everywhere in E. For each $x \in A$ let

$$f(x) = \begin{cases} F'(x) & \text{if } x \in E \text{ and } F \text{ is derivable at } x, \\ 0 & \text{otherwise.} \end{cases}$$

The indefinite Lebesgue integral $G = \int f(x)\,dx$ is an additive function on the family of all subcells of A that is derivable to f almost everywhere in A ([7, Chapter IV, Theorem 6.3]).

Choose an $\varepsilon > 0$ and define an e-gage δ on E as follows: if $x \in E$ is such that $F'(x) = G'(x) = f(x)$, find a $\delta(x) > 0$ so that $|F(B) - G(B)| < \varepsilon|B|$ for each cell $B \subset A$ with $s(B) > \alpha$, $x \in B$ and $d(B) < \delta(x)$; otherwise, let $\delta(x) = 0$. For each α-shapely partition $P = \{(A_1, x_1), \ldots, (A_p, x_p)\}$ in A anchored in E that is δ-fine, we obtain

$$\begin{aligned} \sum_{i=1}^{p} |F(A_i)| &< \sum_{i=1}^{p} [\varepsilon|A_i| + |G(A_i)|] \le \sum_{i=1}^{p} [\varepsilon|A_i| + \int_{A_i} |f(x)|\,dx] \\ &\le \varepsilon|A| + \int_A |f(x)|\,dx = \varepsilon|A| + \int_E |\underline{F}(x)|\,dx. \end{aligned}$$

Consequently

$$F^{\alpha,\delta}(E) \le \varepsilon|A| + \int_E |\underline{F}(x)|\,dx,$$

and a contradiction to (1) follows from the arbitrariness of ε.

Conversely, assume $F^{e*}(E) < \int_E |\underline{F}(x)|\,dx$. Let $Y = \{x \in E : \underline{F}(x) = -\infty\}$ and let n be a positive integer. For each $x \in Y$ there is an $\alpha_x > 0$ such that given $\theta > 0$, we can find a cell $B \subset A$ with $x \in B$, $s(B) > \alpha_x$, $d(B) < \theta$, and $F(B) < -n|B|$. Fix a positive integer j, let $Y_j = \{x \in Y : \alpha_x > 1/j\}$ and find an e-gage δ on E so that $F^{1/j,\delta}(E) < \int_E |\underline{F}(x)|\,dx$. The family $\mathcal{B}$ of all cells $B \subset A$ such that $s(B) > 1/j$, $d(B) < \delta(x)$ for an $x \in B \cap Y_j$, and $F(B) < -n|B|$ is a Vitali cover of $Y_j - N_\delta$. Noting that N_δ is a negligible set and using Vitali's covering theorem ([7, Chapter IV, Theorem 3.1]), we can find a $(1/j)$-shapely partition $P = \{(B_1, y_1), \ldots, (B_q, y_q)\}$ in A anchored in E, such that P is δ-fine, $F(B_i) < -n|B_i|$ for $i = 1, \ldots, q$, and $\sum_{i=1}^{q} |B_i| > |Y_j|/2$. It follows that

$$|Y_j| < 2\sum_{i=1}^{q} |B_i| < \frac{2}{n}\sum_{i=1}^{q} |F(B_i)| \le \frac{2}{n} F^{1/j,\delta}(E),$$

and so Y_j is negligible by the arbitrariness of n. Since $Y = \bigcup_{j=1}^{\infty} Y_j$ is also negligible, F is derivable almost everywhere in E by Ward's theorem.

Applying Luzin's theorem ([7, Chapter III, Theorem 7.1]), construct an increasing sequence $\{E_n\}$ of closed subsets of E such that F' is defined and continuous on each E_n, and $|E - \bigcup_{n=1}^{\infty} E_n| = 0$. Select a set $Z = E_n$ so that

$$F^{e*}(E) < \int_Z |\underline{F}(x)|\,dx = \int_Z |F'(x)|\,dx,$$

and choose an $\varepsilon > 0$. There is a positive function σ on Z such that $|F'(x) - F'(y)| < \varepsilon$ for each $x, y \in Z$ with $d(\{x,y\}) < \sigma(x)$. Given $x \in Z$, find a $\beta_x > 0$ so that

$$\big|F(C) - F'(x)|C|\big| < \varepsilon|C|$$

for each cell $C \subset A$ of sufficiently small diameter for which $x \in C$ and $s(C) > \beta_x$. For $j = 1, 2, \ldots$, let $Z_j = \{x \in Z : \beta_x > 1/j\}$, and select a measurable set X_j so that $Z_j \subset X_j \subset Z$ and $|X_j| = |Z_j|$. Replacing X_j by $\bigcap_{i=j}^{\infty} X_i$, we may assume $\{X_j\}$ is an increasing sequence. Since $Z = \bigcup_{j=1}^{\infty} X_j$, there is a positive integer k such that

$$F^{e*}(E) < \int_{X_k} |F'(x)|\, dx.$$

Find an e-gage Δ on E so that

$$F^{1/k,\Delta}(E) < \int_{X_k} |F'(x)|\, dx. \tag{2}$$

With no loss of generality, we may assume $\Delta(x) \le \sigma(x)$ for each $x \in Z$. The family $\mathcal{C}$ of all cells $C \subset A$ with $s(C) > 1/k$ and such that $d(C) < \Delta(x)$ and

$$\big|F(C) - F'(x)|C|\big| < \varepsilon|C|$$

for an $x \in C \cap Z_k$ is a Vitali cover of $Z_k - N_\Delta$. As N_Δ is a negligible set, according to Vitali's covering theorem, there is a disjoint sequence $\{C_i\}$ in $\mathcal{C}$ such that

$$|X_k - \bigcup_{i=1}^{\infty} C_i| = |Z_k - \bigcup_{i=1}^{\infty} C_i| = 0.$$

For $i = 1, 2, \ldots$, select a $z_i \in C_i \cap Z_k$ so that $d(C_i) < \Delta(z_i) \le \sigma(z_i)$ and

$$\big|F(C_i) - F'(z_i)|C_i|\big| < \varepsilon|C_i|,$$

and observe that

$$\begin{aligned}\int_{X_k} |F'(x)|\, dx &\le \sum_{i=1}^{\infty} \int_{C_i \cap Z} |F'(x)|\, dx \le \sum_{i=1}^{\infty} \Big(\sup_{x \in C_i \cap Z} |F'(x)|\Big)|C_i| \\ &\le \sum_{i=1}^{\infty} \big(|F'(z_i)| + \varepsilon\big)|C_i| < \sum_{i=1}^{\infty} \big(|F(C_i)| + 2\varepsilon|C_i|\big) \\ &\le \sum_{i=1}^{\infty} |F(C_i)| + 2\varepsilon|A|.\end{aligned}$$

Since, for each positive integer s, the collection $\{(C_1, z_1), \ldots, (C_s, z_s)\}$ is a partition in A anchored in E that is Δ-fine, we see that $\sum_{i=1}^{\infty} |F(C_i)| \le F^{1/k,\Delta}(E)$. Thus

$$\int_{X_k} |F'(x)|\, dx \le F^{1/k,\Delta}(E) + 2\varepsilon|A|,$$

and a contradiction to (2) follows from the arbitrariness of ε.

COROLLARY 3.4. *An additive function F on the family of all subcells of a cell $A \subset \mathbf{R}^m$ is derivable almost everywhere in A if and only if it is* BV_{e*}.

PROOF. If $A = \bigcup_{n=1}^{\infty} E_n$ and $F^{e*}(E_n) < +\infty$ for $n = 1, 2, \dots$, then it follows from Theorem 3.3 that $|\underline{F}(x)| < +\infty$ for almost all $x \in A$. By Ward's theorem F is derivable almost everywhere in A.

Conversely, assume F is derivable in $A - E_0$ where E_0 is a negligible set, and let $E_n = \{x \in A - E_0 : |F'(x)| < n\}$ for $n = 1, 2, \dots$. Then $E = \bigcup_{n=0}^{\infty} E_n$ and, by Theorem 3.3, we have

$$F^{e*}(E_n) = \int_{E_n} |F'(x)|\, dx \leq n|E_n| < +\infty$$

for $n = 1, 2, \dots$. Since $F^*(E_0) = 0$, the corollary is proved.

4. Descriptive definition. Let $A \subset \mathbf{R}^m$ be a cell. We say that an additive function on the family of all subcells of A is *continuous* whenever the unique extension of F to an additive function on the family of all subfigures of A is continuous in the sense of Definition 2.4. With this convention in mind, we proceed to characterize those continuous additive functions on the family of all subcells of A which are indefinite g-integrals.

PROPOSITION 4.1. *Let f be a g-integrable function on a cell $A \subset \mathbf{R}^m$. If F is the indefinite g-integral of f, then F is* BV_*.

PROOF. With no loss of generality we may assume that f is a real-valued function defined on the whole of A. Choose an $\alpha > 0$, and for $n = 1, 2, \dots$, let $E_n = \{x \in A : |f(x)| < n\}$. According to Definition 2.5, there is a gage δ on A such that

$$\sum_{i=1}^{p} \bigl| f(x_i)|A_i| - F(A_i) \bigr| < 1$$

for each α-shapely δ-fine partition $P = \{(A_1, x_1), \dots, (A_p, x_p)\}$ in A. If P is anchored in E_n, then

$$\sum_{i=1}^{p} |F(A_i)| < \sum_{i=1}^{p} |f(x_i)| \cdot |A_i| + 1 \leq n|A| + 1,$$

and hence $F^{\alpha,\delta}(E_n) \leq n|A| + 1$. From the arbitrariness of α, we conclude

$$F^*(E_n) \leq n|A| + 1 < +\infty,$$

and as $A = \bigcup_{n=1}^{\infty} E_n$, the proposition is proved.

Since each BV_* function is a BV_{e*} function, Corollary 3.4 together with Proposition 4.1 imply that each indefinite g-integral is derivable almost everywhere. A stronger statement, provided by the next theorem, is proved independently.

THEOREM 4.2. *Let f be a g-integrable function on a cell $A \subset \mathbf{R}^m$, and let F be the indefinite g-integral of f. For almost all $x \in A$ the function F is derivable at x and $F'(x) = f(x)$.*

PROOF. We may assume f is a real-valued function. Denote by E the set of all $x \in A$ for which either F is not derivable at x or $F'(x) \neq f(x)$. Given $x \in E$, we can find an $\alpha_x > 0$ so that for each $\beta > 0$ there is a cell $B \subset A$ with $x \in B, s(B) > \alpha_x, d(B) < \beta$, and

$$\left|f(x)|B| - F(B)\right| \geq \alpha_x |B|.$$

Fix an integer $n \geq 2$, let $E_n = \{x \in E : \alpha_x \geq 1/n\}$ and choose an $\varepsilon > 0$. There is a gage δ on A such that

$$\sum_{i=1}^{p} \left|f(x_i)|A_i| - F(A_i)\right| < \frac{\varepsilon}{n}$$

for each $(1/n)$-shapely δ-fine partition $\{(A_1, x_1), \dots, (A_p, x_p)\}$ in A. The family $\mathcal{B}$ of all cells $B \subset A$ with $s(B) > 1/n$ and such that $d(B) < \delta(x)$ and

$$\left|f(x)|B| - F(B)\right| \geq \frac{1}{n}|B|$$

for an $x \in B \cap E_n$ is a Vitali cover of $E_n - N_\delta$. As N_δ is a negligible set, by Vitali's covering theorem, there is a disjoint sequence $\{B_k\}$ in $\mathcal{B}$ such that $|E_n - \bigcup_{k=1}^{\infty} B_k| = 0$. In each $B_k \cap E_n$ select a point x_k so that $d(B_k) < \delta(x_k)$ and

$$\left|f(x_k)|B_k| - F(B_k)\right| \geq \frac{1}{n}|B_k|.$$

Observe that for $p = 1, 2, \dots$, the collection $\{(B_1, x_1), \dots, (B_p, x_p)\}$ is a $(1/n)$-shapely δ-fine partition in A. Thus

$$\sum_{k=1}^{p} |B_k| \leq n \sum_{k=1}^{p} \left|f(x_k)|B_k| - F(B_k)\right| < \varepsilon$$

for all positive integers p, and consequently $|E_n| \leq \sum_{k=1}^{\infty} |B_k| \leq \varepsilon$. The arbitrariness of ε implies that E_n is negligible, and so is the set $E = \bigcup_{n=2}^{\infty} E_n$.

DEFINITION 4.3. A continuous additive function on the family of all subcells of a cell $A \subset \mathbf{R}^m$ is called AC_* whenever F^* is absolutely continuous with respect to the Lebesgue measure.

It is easy to see that the AC_* functions defined above coincide with those of [6, Definition 11.6.3]. We are now ready to provide a full descriptive definition of the g-integral.

THEOREM 4.4. *Let F be a continuous additive function on the family of all subcells of a cell $A \subset \mathbf{R}^m$. The following conditions are equivalent:*

1. *F is AC_* and BV_{**};*
2. *F is AC_* and BV_{e*};*
3. *F' exists almost everywhere in A, and F is its indefinite g-integral.*

PROOF ($2 \Rightarrow 3$). By Corollary 3.4, F is derivable almost everywhere in A. Denote by E the negligible set of all $x \in A$ at which F is not derivable, and let

$$f(x) = \begin{cases} F'(x) & \text{if } x \in A - E, \\ 0 & \text{if } x \in E. \end{cases}$$

It suffices to show F is the indefinite g-integral of f. To this end, choose an $\varepsilon > 0$, and find a gage α on E so that $\sum_{i=1}^{p} |F(A_i)| < \varepsilon/2$ for each ε-shapely partition $\{(A_1, x_1), \ldots, (A_p, x_p)\}$ anchored in E that is α-fine; such an α exists, since F is AC$_*$ and E is a negligible set. On $A - E$ there is a positive function β such that

$$\left| f(x)|B| - F(B) \right| < \frac{\varepsilon}{2|A|}|B|$$

for each $x \in A - E$ and each cell $B \subset A$ with $s(B) > \varepsilon$, $x \in B$ and $d(B) < \beta(x)$. Define a gage δ on A by setting

$$\delta(x) = \begin{cases} \alpha(x) & \text{if } x \in E, \\ \beta(x) & \text{if } x \in A - E, \end{cases}$$

and choose an ε-shapely δ-fine partition $\{(C_1, z_1), \ldots, (C_q, z_q)\}$ in A. Then

$$\sum_{i=1}^{q} \left| f(z_i)|C_i| - F(C_i) \right| < \sum_{z_i \in E} |F(C_i)| + \frac{\varepsilon}{2|A|} \sum_{z_i \notin E} |C_i| \leq \varepsilon,$$

which implies F is, indeed, the indefinite g-integral of f.

($3 \Rightarrow 1$). It follows from Proposition 4.1 that F is BV$_*$. To show F is also AC$_*$, select a negligible set $E \subset A$ and an $\varepsilon > 0$. Making E larger, we may assume F is derivable everywhere in $A - E$. Then F is the indefinite g-integral of the function f defined in the proof of ($2 \Rightarrow 3$). Find a gage δ on A so that

$$\sum_{i=1}^{p} \left| f(x_i)|A_i| - F(A_i) \right| < \varepsilon$$

for each ε-shapely δ-fine partition $P = \{(A_1, x_1), \ldots, (A_p, x_p)\}$ in A. If such a partition P is anchored in E, we obtain $\sum_{i=1}^{p} |F(A_i)| < \varepsilon$, and consequently $F^{\varepsilon,\delta}(E) \leq \varepsilon$. This and the arbitrariness of ε imply $F^*(E) = 0$.

As the remaining implication ($1 \Rightarrow 2$) is obvious, the proof of the theorem is completed.

REMARK 4.5. From the one dimensional case $m = 1$ in Theorem 4.4 it follows that a function F is ACG$_*$ on an interval in the sense of Saks [7] if and only if F is BV$_*$ and AC$_*$ in our sense. It can be proved too that a continuous function F is VBG$_*$ on an interval in the sense of Saks if and only if F is BV$_*$ in our sense.

Let $A \subset \mathbf{R}^m$ be a cell and let $v: A \longrightarrow \mathbf{R}^m$ be a continuous vector field. The *flux* of v, *i.e.*, the map $B \longmapsto \int_{\partial B} v \cdot n_B \, d\mathcal{H}$ where n_B denotes the unit exterior normal of a cell $B \subset A$, is an additive continuous function defined on the family of all subcells of A ([6, Propositions 8.3.2 and 11.2.8]). The divergence theorem for the g-integral ([6, Theorem 11.7.5]) yields immediately an interesting corollary of Theorem 4.4. As usual, we let $|x - y| = d(\{x, y\})$ for all $x, y \in \mathbf{R}^m$.

COROLLARY 4.6. *Let F be the flux of a continuous vector field v defined on a cell $A \subset \mathbf{R}^m$. If there is a thin set $T \subset A$ such that*

$$\limsup_{y \to x} \frac{|v(y) - v(x)|}{|y - x|} < +\infty$$

for each $x \in A - (T \cup \partial A)$, then F is AC_* *and* BV_*.

The following proposition and example illuminate the relationship between the critical variation and essential critical variation.

PROPOSITION 4.7. *Let F be an additive function on the family of all subcells of a cell $A \subset \mathbf{R}^m$. If $F^*(N) = 0$ for each negligible set $N \subset A$, then $F^* = F^{e*}$. In particular, $F^* = F^{e*}$ whenever F is* AC_*.

PROOF. Assume $F^{e*}(E) < F^*(E)$ for a set $E \subset A$, and find an $\alpha > 0$ and an e-gage ω on E so that $F^{\alpha,\omega}(E) < F^{\alpha,\delta}(E)$ for each gage δ on E. Since $N = N_\omega$ is a negligible set, given $\varepsilon > 0$, there is a gage δ_N on N such that $F^{\alpha,\delta_N}(N) < \varepsilon$. Define a gage δ on E by setting

$$\delta(x) = \begin{cases} \omega(x) & \text{if } x \in E - N, \\ \delta_N(x) & \text{if } x \in N, \end{cases}$$

and observe that

$$F^{\alpha,\delta}(E) \le F^{\alpha,\omega}(E) + F^{\alpha,\delta_N}(N) < F^{\alpha,\omega}(E) + \varepsilon.$$

A contradiction follows from the arbitrariness of ε. Thus $F^*(E) \le F^{e*}(E)$, and as the reverse inequality is obvious, the proposition is proved.

EXAMPLE 4.8. Let C be the Cantor ternary set in $A = [0, 1]$, and let F be the Cantor function on A (see [6, Example 5.3.11]) viewed as an additive function on the family of all subcells of A. For each $x \in A$, let $\omega(x)$ be the distance from x to C. As C is a closed negligible set, the function $\omega: x \longmapsto \omega(x)$ is an e-gage on A. It follows that $F^{e*}(A) = 0$.

On the other hand, the Cantor function on A is the canonical example of an increasing continuous function which is not absolutely continuous. Thus, by [6, Propositions 6.4.6 and 6.4.5], the function F is not AC_*. In particular, $F^*(A) > 0$. An application of Proposition 4.9 below actually shows that $F^*(A) = F^*(C) = 1$.

PROPOSITION 4.9. *Let F be a continuous additive function on the family of all subcells of a cell $A \subset \mathbf{R}^m$. Then $F^*(A)$ is the usual variation of F on A.*

PROOF. If V_F denotes the usual variation of F, then $F^{\alpha,\delta}(A) \le V_F(A)$ for each $\alpha > 0$ and each gage δ on A. We conclude $F^*(A) \le V_F(A)$, and proceeding towards a contradiction, assume $F^*(A) < V_F(A)$. There are nonoverlapping cells $D_1, \ldots, D_n$ such that $\bigcup_{k=1}^n D_k = A$ and $F^*(A) < \sum_{k=1}^n |F(D_k)|$. Choose a positive $\alpha < 1$ and find a gage δ on A so that

$$F^{\alpha,\delta}(A) < \sum_{k=1}^{n} |F(D_k)|.$$

Given $\varepsilon > 0$, it follows from [6, Lemma 11.3.4 and Proposition 11.3.7] that in each D_k there is an α-shapely δ-fine partition $P_k = \{(A_1^k, x_1^k), \dots, (A_{p_k}^k, x_{p_k}^k)\}$ such that

$$\sum_{i=1}^{p_k} |F(A_i^k)| \geq \left|F\Big(\bigcup_{i=1}^{p_k} A_i^k\Big)\right| > |F(D_k)| - \frac{\varepsilon}{n}.$$

Clearly $\bigcup_{k=1}^{n} P_k$ is an α-shapely δ-fine partition in A, and so

$$F^{\alpha,\delta}(A) \geq \sum_{k=1}^{n} \sum_{i=1}^{p_k} |F(A_i^k)| > \sum_{k=1}^{n} |F(D_k)| - \varepsilon.$$

Since ε is arbitrary, this is a contradiction.

In view of [6, Propositions 6.4.6 and 6.4.5], the next corollary follows immediately from Proposition 4.9. Together with Theorem 4.4, it illustrates the difference between the gage and Lebesgue integrals.

COROLLARY 4.10. *Let F be a continuous additive function on the family of all subcells of a cell $A \subset \mathbf{R}^m$. The following conditions are equivalent:*

1. *F is* AC_* *and $F^*(A) < +\infty$;*
2. *F is* AC_* *and $F^{e*}(A) < +\infty$;*
3. *F' exists almost everywhere in A, and F is its indefinite Lebesgue integral.*

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Canad. Math. Bull. Vol. **39** (4), 1996 pp. 402–407

UNE CARACTÉRISATION DES POLYNÔMES PRENANT DES VALEURS ENTIÈRES SUR TOUS LES NOMBRES PREMIERS

JEAN-LUC CHABERT

RÉSUMÉ. Nous donnons une caractérisation des polynômes à coefficients rationnels prenant des valeurs entières sur tous les nombres premiers: pour tester un polynôme donné de degré n, il suffit de considérer ses valeurs sur les entiers de 1 à $2n-1$.

ABSTRACT. We give a characterization of polynomials with rational coefficients which take integral values on the prime numbers: to test a polynomial of degree n, it is enough to consider its values on the integers from 1 to $2n-1$.

NOTATIONS. Pour toute partie E de $\mathbb{Z}$ et tout sous-anneau A de $\mathbb{Q}$, nous notons $\mathrm{Int}(E,A)$ la A-algèbre formée des polynômes prenant sur E leurs valeurs dans A (*cf.* [1]), c'est-à-dire :

$$\mathrm{Int}(E,A)=\{f(X)\in\mathbb{Q}[X]\mid f(k)\in A \text{ pour tout } k\in E\}.$$

En particulier, $\mathrm{Int}(\mathbb{Z},\mathbb{Z})$ correspond à l'anneau bien connu, noté $\mathrm{Int}(\mathbb{Z})$, des polynômes à valeurs entières sur $\mathbb{Z}$ (*cf.* par exemple [3]).

Désignant par $\mathcal{P}$ l'ensemble des nombres premiers, nous nous intéressons ici à l'anneau $\mathrm{Int}(\mathcal{P},\mathbb{Z})=\{f(x)\in\mathbb{Q}[X]\mid f(\mathcal{P})\subset\mathbb{Z}\}$ des polynômes à valeurs entières sur les nombres premiers.

Il est classique qu'un polynôme $f(x)$ de degré n est à valeurs entières sur les entiers, c'est-à-dire appartient à $\mathrm{Int}(\mathbb{Z})$, si et seulement si $f(0),f(1),\ldots,f(n)$ sont des entiers. Cela résulte notamment de ce que les polynômes $\binom{X}{n}=\frac{X(X-1)\cdots(X-n+1)}{n!}$ forment une base du $\mathbb{Z}$-module $\mathrm{Int}(\mathbb{Z})$ et qu'ils vérifient $\binom{k}{n}=0$ pour $0\le k<n$ et $\binom{n}{n}=1$.

Notons que $\mathrm{Int}(\mathcal{P},\mathbb{Z})$ contient strictement $\mathrm{Int}(\mathbb{Z})$: considérer le polynôme $\frac{(X-1)(X-2)(X-3)}{24}$. Rappelons aussi que $\mathrm{Int}(\mathcal{P},\mathbb{Z})$ est un $\mathbb{Z}$-module libre dont on sait construire une base $\big(C_n(X)\big)_{n\in\mathbb{N}}$ où $\deg(C_n)=n$ (*cf.* [2]). Toutefois il n'existe pas de base de $\mathrm{Int}(\mathcal{P},\mathbb{Z})$ possédant une propriété analogue à celle de la base $\big(\binom{X}{n}\big)_{n\in\mathbb{N}}$ de $\mathrm{Int}(\mathbb{Z})$, sauf en ce qui concerne les petits degrés car on peut choisir pour premiers éléments de la base les polynômes

$$C_0=1,\ C_1=X-1,\ C_2=\frac{(X-1)(X-2)}{2}\ \text{et}\ C_3=\frac{(X-1)(X-2)(X-3)}{24},$$

Reçu par les éditeurs le September 27, 1995.
Classification (AMS) par sujet : Primary: 11C08; Secondary: 11A41, 11Y16.
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et on a donc

$$C_0(1) = C_1(2) = C_2(3) = C_3(5) = 1.$$

Ainsi, en posant $p_0 = 1, p_1 = 2, p_2 = 3$ et $p_3 = 5$, on voit qu'un polynôme $f(X)$ de degré $n \leq 3$ appartient à $\mathrm{Int}(\mathcal{P}, \mathbb{Z})$ si et seulement si $f(p_k)$ appartient à $\mathbb{Z}$ pour $0 \leq k \leq n$.

Essayer cependant d'utiliser une base pour tester l'appartenance à $\mathrm{Int}(\mathcal{P}, \mathbb{Z})$ d'un polynôme $f(X)$ de degré élevé risque d'être très long car, d'une part la construction de la base $\big(C_n(X)\big)$ nécessite un calcul pour chaque degré n, calcul qui s'avère n'être ni récurrent, ni récursif, d'autre part le calcul des coefficients de $f(X)$ dans cette base serait vite fastidieux.

Nous montrons toutefois dans cet article que, pour savoir si un polynôme de degré n est à valuers entières sur les nombres premiers, il suffit de calculer au plus $2n$ valeurs de ce polynôme. Très précisément :

1. THÉORÈME. *Un polynôme $f(X)$ à coefficients rationnels de degré $n > 0$ est à valeurs entières sur les nombres premiers si et seulement si :*

(i) pour tout nombre premier $p \leq n+1$, $f(p)$ est un entier,

(ii) pour tout entier naturel $h \leq 2n-1$, $h^{2n-5}f(h)$ est un entier.

La démonstration de ce théorème nécessite un certain nombre de résultats préliminaires. Rappelons tout d'abord une conséquence du théorème de Dirichlet sur les nombres premiers.

2. PROPOSITION [2]. *Soient $f \in \mathrm{Int}(\mathcal{P}, \mathbb{Z})$, $p \in \mathcal{P}$ et $h \in \mathbb{Z}$. Si p ne divise pas h, alors p ne divise pas le dénominateur de $f(h)$. Autrement dit, pour tout nombre premier p, on a :*

$$\mathrm{Int}(\mathcal{P}, \mathbb{Z}) \subset \mathrm{Int}(\mathbb{Z} \setminus p\mathbb{Z}, \mathbb{Z}_{(p)}).$$

En effet, soit ν un entier tel que $p^{\nu}f(X)$ appartienne à $\mathbb{Z}_{(p)}[X]$ et soit q un nombre premier apparaissant dans la suite $(h + np^{\nu})_{n\in\mathbb{N}}$. Alors $p^{\nu}\big(f(h) - f(q)\big) \in (h-q)\mathbb{Z}_{(p)} \subset p^{\nu}\mathbb{Z}_{(p)}$ et donc $f(h) \in \mathbb{Z}_{(p)}$.

3. PROPOSITION. *Pour tout $p \in \mathcal{P}$,* $\mathrm{Int}(\mathcal{P}, \mathbb{Z}_{(p)}) = \mathrm{Int}\big(\{p\} \cup (\mathbb{Z} \setminus p\mathbb{Z}), \mathbb{Z}_{(p)}\big)$.

En effet, $\mathcal{P} \subset \{p\} \cup (\mathbb{Z} \setminus p\mathbb{Z})$.

On a par ailleurs la formule immédiate :

$$\mathrm{Int}(\mathcal{P}, \mathbb{Z}) = \cap_{p\in\mathcal{P}} \mathrm{Int}(\mathcal{P}, \mathbb{Z}_{(p)}).$$

NOTATION. Désormais, sauf mention contraire, p désigne un nombre premier fixé et ν_p la valuation associée. Pour tout entier $k \geq 1$, posons $u_k = k + [\frac{k-1}{p-1}]$. La suite (u_k) est alors la suite croissante des entiers naturels non multiples de p et on a $\mathbb{Z} \setminus p\mathbb{Z} = \{\pm u_k \mid k \in \mathbb{N}^*\}$.

4. LEMME. *Pour tout $n \in \mathbb{N}^*$ et pour tout $m \in \mathbb{Z} \setminus p\mathbb{Z}$, on a :*

$$\begin{aligned} v_p\Big(\prod_{1\leq k\leq n}(m-u_k)\Big) &\geq v_p\Big(\prod_{1\leq k\leq n}(u_{n+1}-u_k)\Big) \\ &= \sum_{s\geq 0}\Big[\frac{n}{p^s(p-1)}\Big]. \end{aligned}$$

DÉMONSTRATION. Comme $v_p(u_{n+1}) = 0$, en ajoutant des termes, on obtient

$$\begin{aligned} v_p\Big(\prod_{1\leq k\leq n}(u_{n+1}-u_k)\Big) &= v_p\Big(\prod_{0\leq k<u_{n+1}}(u_{n+1}-k)\Big) \\ &= v_p\big((u_{n+1})!\big). \end{aligned}$$

De même,

$$\begin{aligned} v_p\Big(\prod_{1\leq k\leq n}(m-u_k)\Big) &= v_p\Big(\prod_{0\leq k<u_{n+1}}(m-k)\Big) \\ &= v_p\Big(\frac{m!}{(m-u_{n+1})!}\Big). \end{aligned}$$

L'inégalité résulte alors de ce que $\binom{m}{u_{n+1}} \in \mathbb{Z}$.

Pour l'égalité, on sait que

$$v_p\big((u_{n+1})!\big) = \sum_{s\geq 1}\Big[\frac{u_{n+1}}{p^s}\Big];$$

il suffit donc de vérifier que

$$\Big[\frac{u_{n+1}}{p}\Big] = \Big[\frac{n}{p-1}\Big],$$

puisqu'on aura alors

$$\Big[\frac{u_{n+1}}{p^s}\Big] = \Big[\frac{n}{p^{s-1}(p-1)}\Big].$$

Soient q et r tels que

$$n = q(p-1) + r \text{ avec } 0 \leq r < p-1.$$

Alors

$$\begin{aligned} u_{n+1} &= n+1+\Big[\frac{n}{p-1}\Big] \\ &= qp + (r+1) \text{ avec } 1 \leq r+1 < p. \end{aligned}$$

NOTATION. Pour tout entier $n \geq 0$, posons $\omega_p(n) = \sum_{s\geq 0}[\frac{n-1}{p^s(p-1)}]$.

Cette fonction $\omega_p(n)$ rappelle la fonction $w_p(n) = \sum_{s\geq 1}[\frac{n}{q^s}]$ qui intervient dans la détermination des polynômes à valeurs entières sur un corps de nombres (*cf.* [4] et [5]).

Posons aussi $D_0(X) = 1$ et, pour $n \geq 1$:

$$D_n(X) = p^{-\omega_p(n+1)} \prod_{1\leq k\leq n}(X-u_k).$$

Il résulte du Lemme 4 que les polynômes $D_n(X)$ appartiennent à $\text{Int}(\mathbb{Z}\setminus p\mathbb{Z}, \mathbb{Z})$ et qu'ils forment une base du $\mathbb{Z}_{(p)}$-module $\text{Int}(\mathbb{Z}\setminus p\mathbb{Z}, \mathbb{Z}_{(p)})$. En effet, pour $k = 1, \ldots, n, D_n(u_k) = 0$ et $D_n(u_{n+1})$ est inversible dans $\mathbb{Z}_{(p)}$. Par suite :

5. Lemme. *(i) Un polynôme $f(X) \in \mathbb{Q}[X]$ de degré n appartient à* $\mathrm{Int}(\mathbb{Z} \setminus p\mathbb{Z}, \mathbb{Z}_{(p)})$ *si et seulement si $f(u_k) \in \mathbb{Z}_{(p)}$ pour $k = 1, \ldots, n+1$.*

(ii) Si un polynôme $f(X) \in \mathbb{Q}[X]$ de degré n appartient à $\mathrm{Int}(\mathbb{Z} \setminus p\mathbb{Z}, \mathbb{Z}_{(p)})$*, alors $p^{\omega_p(n+1)}f(X)$ appartient à $\mathbb{Z}_{(p)}[X]$.*

De façon analogue, considérons les polynômes $C_0(X) = 1$ et, pour $n \geq 1$:

$$C_n(X) = (X - p)D_{n-1}(X).$$

Les polynômes $C_n(X)$ appartiennent à $\mathrm{Int}(\mathcal{P}, \mathbb{Z})$ et forment une base du $\mathbb{Z}_{(p)}$-module $\mathrm{Int}\big(\{p\} \cup (\mathbb{Z} \setminus p\mathbb{Z}), \mathbb{Z}_{(p)}\big)$ (*cf.* aussi [2]). Par suite, compte tenu de la Proposition 3 :

6. Lemme. *(i) Un polynôme $f(X) \in \mathbb{Q}[X]$ de degré n appartient à* $\mathrm{Int}(\mathcal{P}, \mathbb{Z}_{(p)})$ *si et seulement si $f(p) \in \mathbb{Z}_{(p)}$ et $f(u_k) \in \mathbb{Z}_{(p)}$ pour $k = 1, \ldots, n$.*

(ii) Si un polynôme $f(X) \in \mathbb{Q}[X]$ de degré n appartient à $\mathrm{Int}(\mathcal{P}, \mathbb{Z}_{(p)})$*, alors $p^{\omega_p(n)}f(X)$ appartient à $\mathbb{Z}_{(p)}[X]$.*

Il nous faut maintenant majorer la quantité $\omega_p(n)$.

7. Lemme. *Pour tout entier $n \geq 1$, on a :*

$$\omega_p(n) \leq \frac{(n-1)p}{(p-1)^2} - \frac{1}{p-1}.$$

En particulier, pour $n \geq 3$, $\omega_2(n) \leq 2n - 3$ et, pour tout $p \geq 3$, $\omega_p(n) \leq 2n - 5$.

Démonstration. On a

$$\begin{aligned}\omega_p(n) &= \sum_{s \geq 0} \left[\frac{n-1}{p^s(p-1)}\right] \\ &= \sum_{0 \leq s \leq r} \left[\frac{n-1}{p^s(p-1)}\right]\end{aligned}$$

où r vérifie

$$p^r \leq \frac{n-1}{p-1} < p^{r+1}.$$

Alors

$$\begin{aligned}\omega_p(n) &\leq \frac{n-1}{p-1} \sum_{0 \leq s \leq r} \frac{1}{p^s} \\ &\leq \frac{n-1}{(p-1)^2}\left(p - \frac{p-1}{n-1}\right) \\ &= \frac{(n-1)p}{(p-1)^2} - \frac{1}{p-1}.\end{aligned}$$

En particulier, pour $p = 2$,

$$\omega_2(n) \leq 2n - 3,$$

et, pour $p \geq 3$,

$$\begin{aligned}\omega_p(n) &\leq \omega_3(n)\\ &\leq \frac{3}{4}(n-1)-\frac{1}{2}\\ &\leq 2n-5.\end{aligned}$$

8. PROPOSITION. *Soit $f(X)$ un polynôme de degré $n \geq 3$. Si $f(X)$ appartient à* $\mathrm{Int}(\mathcal{P},\mathbb{Z})$, *alors $X^{2n-5}f(X)$ appartient à* $\mathrm{Int}(\mathbb{Z})$.

DÉMONSTRATION. Soit $h \in \mathbb{Z}$. Il suffit de montrer que, pour tout $p \in \mathcal{P}$, $h^{2n-5}f(h)$ appartient à $\mathbb{Z}_{(p)}$. Soit p fixé dans $\mathcal{P}$. Si p ne divise pas h alors $f(h)$ appartient à $\mathbb{Z}_{(p)}$ (Proposition 2). Tandis que si p divise h, on a $v_p(h) \geq 1$ et par suite $v_p\big(h^{2n-5}f(h)\big) \geq 2n-5-\omega_p(n)$ (Lemme 6). De sorte que, si $p \neq 2$, on a bien $v_p\big(h^{2n-5}f(h)\big) \geq 0$ (Lemme 7).

Considérons donc le seul cas où l'on ne peut conclure, celui où $v_2(h) \geq 1$ et $v_2\big(f(h)\big) < 0$. Posons $f(X)-f(2)=(X-2)g(X)$; on sait que $2^{2n-3}f(X) \in \mathbb{Z}_{(2)}[X]$ (Lemmes 6 et 7), donc $2^{2n-3}g(X) \in \mathbb{Z}_{(2)}[X]$ puisque $f(2) \in \mathbb{Z}$. Enfin, comme $v_2\big(f(h)\big) = v_2\big((h-2)g(h)\big)$ et que $v_2\big(h(h-2)\big) \geq 3$, on a bien :

$$\begin{aligned}v_2\big(h^{2n-5}f(h)\big) &\geq 2n-6+v_2\big(hf(h)\big)\\ &\geq 2n-6+v_2\big(h(h-2)\big)+v_2\big(g(h)\big)\\ &\geq 0.\end{aligned}$$

Par exemple, $\frac{(X-1)(X-2)(X-3)}{24}$ appartient à $\mathrm{Int}(\mathcal{P},\mathbb{Z})$ et n'appartient pas à $\mathrm{Int}(\mathbb{Z})$, tandis que $X \cdot \frac{(X-1)(X-2)(X-3)}{24} = \binom{X}{4}$ appartient à $\mathrm{Int}(\mathbb{Z})$.

La Proposition 8 met en particulier en évidence le fait qu'un élément de $\mathrm{Int}(\mathcal{P},\mathbb{Z})$ prend toujours des valeurs entières pour $X=+1$ et $X=-1$ (ce qui en fait était clair dès la Proposition 2).

DÉMONSTRATION DU THÉORÈME 1. Les cas $n=1$ ou 2 se vérifiant “à la main”, on suppose $n \geq 3$. La condition est nécessaire : (i) est évident, (ii) est une conséquence de la Proposition 8. Inversement, supposons que $f(X)$ vérifie (i) et (ii) et montrons que, pour tout $p \in \mathcal{P}$, $f(X)$ appartient à $\mathrm{Int}(\mathcal{P},\mathbb{Z}_{(p)})$.

Premier cas : $p \leq n+1$. Alors, pour $k=1,\ldots,n$, $u_k = k+[\frac{k-1}{p-1}] \leq n+[\frac{n-1}{p-1}] \leq 2n-1$. Donc, d'aprés (ii), $u_k^{2n-5}f(u_k) \in \mathbb{Z}$ et par suite $f(u_k) \in \mathbb{Z}_{(p)}$ puisque p ne divise pas u_k. D'après (i), $f(p) \in \mathbb{Z}$ et donc $f(X) \in \mathrm{Int}(\mathcal{P},\mathbb{Z})_{(p)})$ (Lemme 6).

Second cas : $p > n+1$. Alors, pour $k=1,\ldots,n+1$, $u_k \leq n+1+[\frac{n}{p-1}] = n+1$. Donc, là encore, $f(u_k) \in \mathbb{Z}_{(p)}$ et $f(X) \in \mathrm{Int}(\mathbb{Z}\setminus p\mathbb{Z},\mathbb{Z}_{(p)})$ (Lemme 5). Par suite, $p^{\omega_p(n+1)}f(X) \in \mathbb{Z}_{(p)}[X]$; mais $\omega_p(n+1)=0$, donc $f(X) \in \mathbb{Z}_{(p)}[X]$.

REMARQUES. 1° La preuve précédent montre que la condition (ii) du Théorème 1 peut être remplacée par la condition plus faible :

(ii bis) pour tout entier naturel $h \leq 2n-1$, il existe un entier naturel ν tel que $h^{\nu}f(h)$ soit un entier (referee's remark).

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L'exposant explicite $2n - 5$ confère cependant un caractère algorithmique à la caractérisation donnée par le Théorème 1.

2° Les quantités $n + 1$, $2n - 5$ et $2n - 1$ intervenant dans l'énoncé du Théorème 1 sont optimales.

(a) On ne peut remplacer $2 \leq p \leq n + 1$ par $2 \leq p \leq n$ dans la condition (i) : le polynôme $f(X) = \frac{(X-1)(X-2)(X-3)(X-4)}{5}$ vérifie $f(2) = f(3) = 0$ et $X \cdot f(X) \in \mathrm{Int}(\mathbb{Z})$, alors que $f(5) \notin \mathbb{Z}$.

(b) On ne peut diminuer l'exposant $2n-5$ dans la condition (ii) : soit $n = 2^k+1$, alors $f(X) = \frac{1}{2^{2n-3}}(X-2)\prod_{1 \leq k \leq n-1}\big(X-(2k+1)\big)$ appartient à $\mathrm{Int}(\mathcal{P}, \mathbb{Z})$ (Lemme 6), tandis que $2^{2n-5}f(6)$ est impair.

(c) On ne peut remplacer $h \leq 2n - 1$ par $h < 2n - 1$ dans la condition (ii) : le polynôme $f(X) = \frac{(X-1)(X-2)(X-3)(X-5)}{2^5 \cdot 3}$ vérifie $f(2) = f(3) = f(5) = 0$ et $h^3 f(h) \in \mathbb{Z}$ pour $1 \leq h \leq 6$, mais $f(7) \notin \mathbb{Z}$.

3° En revanche, il est possible de diminuer le nombre de valeurs à calculer en différenciant les nombres premiers. Par exemple, en faisant jouer à $p = 2$ un rôle particulier, la condition (ii) peut être remplacée par la condition plus faible :

(ii ter) pour tout $h \in \mathbb{N}$ vérifiant soit $h < \frac{3}{2}n$, soit h est impair et $\frac{3}{2}n \leq h < 2n$, on a $h^{2n-5}f(h) \in \mathbb{Z}$.

En effet, dans la preuve du Théorème 1, si $p > n + 1$, il n'y a rien à changer. Si $3 \leq p \leq n+1$, la preuve marche encore car, pour $k \leq n$, $u_k = k+[\frac{k-1}{p-1}] \leq n+[\frac{n-1}{2}] < \frac{3}{2}n$. Enfin, si $p = 2$, la preuve marche toujours car on utilise seulement les valeurs $f(u_k)$ où u_k est impair.

On peut se poser la question d'une estimation asymptotique du nombre minimal de valeurs à calculer. Cet article montre que ce nombre est compris entre $n + 1$ et $2n - 1$.

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Canad. Math. Bull. Vol. **39** (4), 1996 pp. 408–419

ON STRONGLY NORMAL FUNCTIONS

HUAIHUI CHEN AND PAUL M. GAUTHIER

ABSTRACT. Loosely speaking, a function (meromorphic or harmonic) from the hyperbolic disk of the complex plane to the Riemann sphere is normal if its dilatation is bounded. We call a function strongly normal if its dilatation vanishes at the boundary. A sequential property of this class of functions is proved. Certain integral conditions, known to be sufficient for normality, are shown to be in fact sufficient for strong normality.

1. **Introduction.** Let $\mathbb{C}$ denote the complex plane, let $D = \{z \in \mathbb{C} : |z| < 1\}$, and let $D_r = \{z \in \mathbb{C} : |z| < r\}$. For a meromorphic function f, let

$$f^{\#}(z) = \frac{|f'(z)|}{1 + |f(z)|^2}$$

denote its spherical derivative. We say that a complex-valued function f in D is *normal* if the family $\mathcal{F} = \{f\bigl(\gamma(z)\bigr) : \gamma \in \mathrm{Aut}(D)\}$ is a normal family in the sense of Montel, where $\mathrm{Aut}(D)$ denotes the collection of all one-to-one conformal mappings of D onto itself. It is known that a function f meromorphic in D is normal if and only if

$$\sup_{z\in D}(1 - |z|^2)f^{\#}(z) < \infty, \tag{1}$$

and that a real-valued function h harmonic in D is normal if and only if

$$\sup_{z\in D}(1 - |z|^2)\frac{|\mathrm{grad}\, h(z)|}{1 + h^2(z)} < \infty. \tag{2}$$

For $\gamma \in \mathrm{Aut}(D)$, setting $g(z) = f\bigl(\gamma(z)\bigr)$, we have

$$(1 - |\gamma(z)|^2)f^{\#}\bigl(\gamma(z)\bigr) = (1 - |z|^2)g^{\#}(z).$$

In particular, if f is a meromorphic function automorphic with respect to a Fuchsian group Γ and $\gamma \in \Gamma$, then

$$\bigl(1 - |\gamma(z)|^2\bigr)f^{\#}\bigl(\gamma(z)\bigr) = (1 - |z|^2)f^{\#}(z).$$

Research project supported in part by NSFC(China), NSERC(Canada) and FCAR(Québec).
Received by the editors June 20, 1995.
AMS subject classification: 30D45 .
Key words and phrases: Normal functions, automorphic functions.
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Thus, for an automorphic meromorphic function f, the above condition (1) can be replaced by

$$\sup_{z\in F}(1-|z|^2)f^{\#}(z)<\infty,$$

where F denotes a fundamental region for the Fuchsian group. Also, for an automorphic harmonic function h, the condition (2) can be replaced by

$$\sup_{z\in F}(1-|z|^2)\frac{|\operatorname{grad} h(z)|}{1+h^2(z)}<\infty.$$

For the definitions and general properties of normal functions, see, for example, [5], [6] and [7].

There are some integral conditions for a function to be normal. Ch. Pommerenke [8] proved that an automorphic meromorphic function f in D which satisfies the condition

$$\iint_F\{f^{\#}(z)\}^2\,dx\,dy<\infty, \tag{3}$$

is normal, where F denotes a fundamental region for the corresponding Fuchsian group. In particular, a meromorphic function f in D with the property that

$$\iint_D\{f^{\#}(z)\}^2\,dx\,dy<\infty \tag{4}$$

is normal. R. Aulaskari, W. K. Hayman, and P. Lappan [1] proved that an automorphic meromorphic function f in D which satisfies

$$I=\iint_F(1-|z|^2)^{p-2}\{f^{\#}(z)\}^p\,dx\,dy<\infty, \tag{5}$$

for some $p>2$, is normal. Furthermore,

$$\sup_{z\in D}\big((1-|z|^2)f^{\#}(z)\big)\le 3\max(I^{1/p},I^{1/(p-2)}).$$

For harmonic functions, Aulaskari and Lappan [3] proved analogous results:

(1) A real-valued harmonic function h with the property

$$\iint_D\left\{\frac{|\operatorname{grad} h(z)|}{1+h^2(z)}\right\}^2dx\,dy<\infty \tag{6}$$

is normal.

(2) If $f=h+i\tilde{h}$ is an automorphic holomorphic function such that

$$\iint_F\left\{\frac{|\operatorname{grad} h(z)|}{1+h^2(z)}\right\}^2dx\,dy<\infty, \tag{7}$$

where F is a fundamental region, then h is normal.

In this paper, we distinguish a subclass of the class of normal meromorphic functions (Definitions 1 and 2). A function in this class is called a strongly normal function. We prove that the conditions (3) and (5) imply not only the normality of the function f but also its strong normality (Theorem 5). An example shows that the inverse is not true (Section 4). Similar definitions and results are given for harmonic functions (Definitions 3 and 4, Theorems 4 and 6). These Theorems strengthen the conclusions of the results mentioned above. Also, we obtain a sequential property of strongly normal functions. It turns out that, for a strongly normal function f, a sequence in the family $\mathcal{F} = \{f(\gamma(z)) : \gamma \in \mathrm{Aut}(D)\}$ contains a subsequence which converges to either $f(\gamma(z))$ with a $\gamma \in \mathrm{Aut}(D)$ or a constant identically (Theorems 1 and 2).

2. **Strongly normal functions.**

Definition 1. A function f meromorphic in D is called a *strongly normal function* if

$$(1 - |z|^2)f^{\#}(z) \longrightarrow 0, \quad z \longrightarrow \partial D.$$

There are many normal meromorphic functions which are not strongly normal. In fact, every meromorphic function automorphic with respect to an infinite group is not strongly normal since $(1 - |z|^2)f^{\#}(z)$ has the same value at equivalent points. For example, the function $\exp[(z-1)/(z+1)]$ cannot be strongly normal.

Definition 2. A function f meromorphic in D and automorphic with respect to a Fuchsian group Γ is called a *strongly normal function with respect to* Γ if

$$(1 - |z|^2)f^{\#}(z) \longrightarrow 0, \quad z \longrightarrow \partial D, \quad z \in F,$$

where F is a fundamental region of Γ.

Definition 3. A real-valued function h harmonic in D is called a *strongly normal function* if

$$(1 - |z|^2)\frac{|\mathrm{grad}\, h(z)|}{1 + h^2(z)} \longrightarrow 0, \quad z \longrightarrow \partial D.$$

Definition 4. A real-valued function h harmonic in D and automorphic with respect to a Fuchsian group Γ is called a *strongly normal function with respect to* Γ if

$$(1 - |z|^2)\frac{|\mathrm{grad}\, h(z)|}{1 + h^2(z)} \longrightarrow 0, \quad z \longrightarrow \partial D, \quad z \in F,$$

where F is a fundamental region for Γ.

As indicated above, even a bounded holomorphic function need not be strongly normal. However, if a function f meromorphic in D satisfies condition (4), that is, if its image has a finite spherical area, counted according to multiplicity, then it turns out that f is strongly normal. Also, condition (6) implies the strong normality of the harmonic function h. For automorphic meromorphic functions, conditions (3) and (5) imply the strong normality, with respect to the corresponding group, of the automorphic meromorphic function f, and condition (7) implies the strong normality of not only f but also the real part h. We will prove these propositions in the next section.

Returning to the normality of the family $\{f(\gamma(z)) : \gamma \in \mathrm{Aut}(D)\}$, we prove a sequential property of strongly normal functions.

THEOREM 1. *If a function f, meromorphic or harmonic in D, is strongly normal, then, for every sequence $\{\gamma_n\} \subset \mathrm{Aut}(D)$, there exists a subsequence $\{\gamma_{n_k}\}$ such that $\{f(\gamma_{n_k}(z))\}$ converges, locally uniformly in D, to either $f(\gamma_0(z))$ with $\gamma_0 \in \mathrm{Aut}(D)$ or a constant identically.*

PROOF. Assume first that f is a meromorphic function. Let

$$\gamma_n(z) = e^{i\theta_n} \cdot \frac{z + z_n}{1 + \bar{z}_n z}.$$

If $z_n \longrightarrow z_0 \in D$, then we may choose a subsequence $\{\gamma_{n_k}\}$ such that $\theta_{n_k} \longrightarrow \theta_0$. Consequently, $\gamma_{n_k} \longrightarrow \gamma_0$, where

$$\gamma_0 = e^{i\theta_0} \cdot \frac{z + z_0}{1 + \bar{z}_0 z},$$

and $f(\gamma_{n_k}(z)) \longrightarrow f(\gamma_0(z))$ locally uniformly in D. Otherwise, we have a subsequence γ_{n_k} such that $z_{n_k} \longrightarrow \partial D$ and $f(z_{n_k}) \longrightarrow w_0$.

Set $f(\gamma_{n_k}(z)) = g_{n_k}(z)$, we have

$$(1 - |z|^2)g^{\#}_{n_k}(z) = (1 - |\gamma_{n_k}(z)|^2)f^{\#}(\gamma_{n_k}(z)).$$

Given a positive number $r' < 1$, the above equality gives

$$\max_{z \in D_{r'}}(1 - |z|^2)g^{\#}_{n_k}(z) = \max_{z \in \gamma_{n_k}(D_{r'})}(1 - |z|^2)f^{\#}(z).$$

It is obvious that, for a fixed r', $\gamma_{n_k}(D_{r'})$ goes to the boundary ∂D of the unit disk as $k \longrightarrow \infty$. Thus,

$$\max_{z \in \gamma_{n_k}(D_{r'})}(1 - |z|^2)f^{\#}(z) \longrightarrow 0, \quad k \longrightarrow \infty,$$

and, consequently,

$$\max_{z \in D_{r'}}(1 - |z|^2)g^{\#}_{n_k}(z) \longrightarrow 0, \quad k \longrightarrow \infty.$$

Note that since $g_{n_k}(0) = f(\gamma_{n_k}(0)) = f(z_{n_k}) \longrightarrow w_0$, we know that $g_{n_k}(z)$ converges to w_0 uniformly for $z \in D_{r'}$. Since r' can be arbitrarily close to 1, this proves Theorem 1 for a meromorphic function f.

The proof for a harmonic function f is exactly a repetition of the above reasoning with $f^{\#}$ and $g^{\#}_{n_k}$ replaced by $|\mathrm{grad}\, f|/(1+f^2)$ and $|\mathrm{grad}\, g_{n_k}|/(1+g^2_{n_k})$ respectively.

The next result asserts that the conclusion of Theorem 1 remains valid for a function which is strongly normal with respect to a Fuchsian group.

THEOREM 2. *If a function f, meromorphic or harmonic in D, is automorphic and strongly normal with respect to a Fuchsian group Γ, then, for every sequence $\{\gamma_n\} \in$ Aut(D), there exists a subsequence $\{\gamma_{n_k}\}$ such that $\{f(\gamma_{n_k}(z))\}$ converges, locally uniformly in D, to either $f(\gamma_0(z))$ with $\gamma_0 \in$ Aut(D) or a constant identically.*

PROOF. We consider the meromorphic case only. Let F be a fundamental region of Γ. Set $z_n = \gamma_n(0)$. Let z'_n be the equivalent point in F of z_n, say $\lambda_n(z_n) = z'_n$. Set $\phi_n(z) = \lambda_n(\gamma_n(z))$, we have

$$f(\gamma_n(z)) = f\Big(\lambda_n\big(\gamma_n(z)\big)\Big) = f(\phi_n(z)).$$

Only the case that $z'_n \longrightarrow \partial D$ needs a more careful discussion.

Fix a positive number r' with $0 < r' < 1$. Now let $\epsilon > 0$ and choose $0 < r_1 < 1$ such that $(1-|z|^2)f^{\#}(z) < \epsilon$ for $z \in F \setminus D_{r_1}$. Choose a positive number r_2 such that $r_1 < r_2 < 1$ and $d(r_1, r_2) = d(0, r')$, where $d(\cdot,\cdot)$ denotes the non-Euclidian distance. The set of points in F which are equivalent to points in D_{r_2} has a positive Euclidian distance to ∂D. Therefore, there exists a positive number r_3 such that $r_2 < r_3 < 1$ and there is no point in $F \setminus D_{r_3}$ equivalent to a point in D_{r_2}.

For sufficiently large n, we have $z'_n \in F \setminus D_{r_3}$. Assume that z' is an arbitrary point in the non-Euclidian disk $\delta_n = \{z \in D : d(z, z'_n) \le d(0, r')\}$. The unique point z'' in F equivalent to z' must lie outside of D_{r_1}. To show this, let $\lambda(z') = z''$ and $\lambda(z'_n) = z''_n$, where $\lambda \in \Gamma$. We have $d(z'', z''_n) = d(z', z'_n) \le d(0, r')$. Therefore, if $z'' \in D_{r_1}$, then $z''_n \in D_{r_2}$ for $d(r_1, r_2) = d(0, r')$. However, z'_n cannot be equivalent to a point in D_{r_2} because of the property of r_3. This proves that $z'' \in F \setminus D_{r_1}$ and consequently,

$$(1-|z'|^2)f^{\#}(z') = (1-|z''|^2)f^{\#}(z'') < \epsilon.$$

Now, set $g_n(z) = f(\phi_n(z))$. Then $g_n(0) = f(z'_n)$. We have

$$\max_{z \in D_{r'}} (1-|z|^2)g^{\#}_n(z) = \max_{z \in \delta_n} (1-|z|^2)f^{\#}(z) < \epsilon.$$

Letting $\epsilon \longrightarrow 0$, we conclude that $g^{\#}_n(z)$ tends to 0 uniformly for $z \in D_{r'}$. Note that we may assume that $g_n(0) = f(z'_n) \longrightarrow w_0$ and r' can be arbitrarily close to 1. We arrive at the conclusion that g_n converges to w_0 locally uniformly in D. This completes the proof.

3. **Main results.**

To prove our main results, we make essential use of the following lemma due to Dufresnoy [4].

Lemma 1. *Let f be a function meromorphic in the disk $D_r = \{z \in \mathbb{C} : |z| < r\}$. If*

$$\iint_{D_r} \{f^{\#}(z)\}^2 \, dx\, dy \le \sigma\pi$$

with $0 \le \sigma < 1$, then

$$f^{\#}(0) \le \frac{1}{r}\left\{\frac{\sigma}{1-\sigma}\right\}^{1/2}.$$

As a consequence of Lemma 1, we have the following.

Lemma 2. *Let h be a real-valued function harmonic in D_r. If*

$$\iint_{D_r} \left\{\frac{|\operatorname{grad} h(z)|}{1+h^2(z)}\right\}^2 dx\, dy \le \sigma\pi,$$

with $0 \le \sigma < 1$, then

$$\frac{|\operatorname{grad} h(0)|}{1+h^2(0)} \le \frac{1}{r}\left\{\frac{\sigma}{1-\sigma}\right\}^{1/2}.$$

Proof. Let $f = h + i\tilde{h}$ be a holomorphic function and $\tilde{h}(0) = 0$. Since

$$f^{\#}(z) = \frac{|f'(z)|}{1+|f(z)|^2} \le \frac{|\operatorname{grad} h(z)|}{1+h^2(z)},$$

we have

$$\iint_{D_r} \{f^{\#}(z)\}^2 \, dx\, dy \le \sigma\pi.$$

Thus, Lemma 1 gives

$$f^{\#}(0) \le \frac{1}{r}\left\{\frac{\sigma}{1-\sigma}\right\}^{1/2}.$$

Since

$$f^{\#}(0) = \frac{|\operatorname{grad} h(0)|}{1+h^2(0)},$$

the conclusion of Lemma 2 follows.

Lemma 3 is a stronger version of Lemma 1 also proved by Dufresnoy.

Lemma 3. *Let f be a function meromorphic in the disk D_r and let A denote the spherical area of $f(D_r)$, counted without consideration of multiplicity. If $A \le \sigma\pi$ with $0 \le \sigma < 1$, then*

$$f^{\#}(0) \le \frac{1}{r}\left\{\frac{\sigma}{1-\sigma}\right\}^{1/2}.$$

Now, we state and prove the main results.

THEOREM 3. *If f is a function meromorphic in D which satisfies condition (4), then f is a strongly normal function.*

PROOF. Fix a small positive number $\sigma < 1$. Let

$$\Delta(z') = \{z \in \mathbb{C} : |z - z'| < 1 - |z'|\}.$$

We have

$$\iint_{\Delta(z')} \{f^{\#}(z)\}^2 \, dx \, dy \leq \sigma\pi,$$

provided that z' is sufficiently close to ∂D. Applying Lemma 1 to the function $f(z)$ and the disk $\Delta(z')$, we obtain

$$f^{\#}(z') \leq \frac{1}{1 - |z'|} \left\{ \frac{\sigma}{1 - \sigma} \right\}^{1/2}.$$

Consequently, for z' sufficiently close to ∂D, we have

$$(1 - |z'|^2) f^{\#}(z') \leq 2 \left\{ \frac{\sigma}{1 - \sigma} \right\}^{1/2}.$$

Letting $\sigma \longrightarrow 0$ we obtain the conclusion of Theorem 3.

The same reasoning, using Lemma 2 instead of Lemma 1, gives the following similar result on harmonic functions.

THEOREM 4. *If h is a real-valued function harmonic in D which satisfies condition (6), then h is a strongly normal function.*

THEOREM 5. *Let f be a function meromorphic in D, which is automorphic with respect to a Fuchsian group Γ. If*

$$\iint_F (1 - |z|^2)^{p-2} \{f^{\#}(z)\}^p \, dx \, dy < \infty$$

for some $p \geq 2$, where F is a fundamental region of Γ, then f is a strongly normal function with respect to Γ.

PROOF. We may assume that $\bar{F}$ is not a compact subset of D. Fix a small positive number σ. There exists a number r_1 such that $0 < r_1 < 1$ and

$$\iint_{F \setminus D_{r_1}} (1 - |z|^2)^{p-2} \{f^{\#}(z)\}^p \, dx \, dy \leq \sigma\pi.$$

Let r_2 be the number such that $r_1 < r_2 < 1$ and $d(r_1, r_2) = 1$. There is a number r_3 with the property that no point in $F \setminus D_{r_3}$ is equivalent to a point in D_{r_2}.

For a point $z' \in F \setminus D_{r_3}$, set $\Delta(z') = \{z \in D : d(z, z') < 1\}$. It is easy to find a measurable set $E' \subset \Delta(z')$ such that no points in E' are equivalent and, for each point $z \in \Delta(z')$, there is a point $\zeta \in E'$ equivalent to z. We have $f\big(\Delta(z')\big) = f(E')$ for f is automorphic. Let E be the subset of F equivalent to E'. The same reasoning as in the proof of Theorem 2 shows that no point of $F \bigcap D_{r_1}$ is equivalent to any point in $\Delta(z')$ and, in particular, it follows that $E \subset F \setminus D_{r_1}$. Denoting by A the spherical area of $f\big(\Delta(z')\big) = f(E')$, without consideration of multiplicity, we have, by Hölder's inequality with respect to the measure $(1-|z|^2)^{-2}\,dx\,dy$,

$$\begin{aligned}
A &\le \iint_{E'} \{f^{\#}(z)\}^2\,dx\,dy = \iint_{E'} \{(1-|z|^2)f^{\#}(z)\}^2 \frac{dx\,dy}{(1-|z|^2)^2} \\
&\le \left\{\iint_{E'} \{(1-|z|^2)f^{\#}(z)\}^p \frac{dx\,dy}{(1-|z|^2)^2}\right\}^{2/p} \left\{\iint_{E'} \frac{dx\,dy}{(1-|z|^2)^2}\right\}^{1-2/p} \\
&= \left\{\iint_{E} \{(1-|z|^2)f^{\#}(z)\}^p \frac{dx\,dy}{(1-|z|^2)^2}\right\}^{2/p} \left\{\iint_{E'} \frac{dx\,dy}{(1-|z|^2)^2}\right\}^{1-2/p} \\
&\le \left\{\iint_{F\setminus D_{r_1}} (1-|z|^2)^{p-2}\{f^{\#}(z)\}^p\,dx\,dy\right\}^{2/p} \left\{\iint_{\Delta(z')} \frac{dx\,dy}{(1-|z|^2)^2}\right\}^{1-2/p} \\
&\le (\sigma\pi)^{2/p} C^{1-2/p},
\end{aligned}$$

where the absolute constant C is the non-Euclidian area of a disk of non-Euclidian radius 1.

Now, let $\gamma \in \mathrm{Aut}(D)$ be a transformation satisfying $\gamma(0) = z'$. Then γ maps some disk D_r onto $\Delta(z')$, where r is the absolute constant such that $d(0, r) = 1$. Applying Lemma 3 to the function $g(z) = f\big(\gamma(z)\big)$, we obtain

$$(1-|z'|^2)f^{\#}(z') = g^{\#}(0) \le \frac{1}{r}\left\{\frac{\sigma^{2/p}(C/\pi)^{1-2/p}}{1-\sigma^{2/p}(C/\pi)^{1-2/p}}\right\}^{1/2}.$$

Since σ may be arbitrarily small, the proof of Theorem 5 is complete.

THEOREM 6. *Let $f = h + i\tilde{h}$ be a function holomorphic in D, which is automorphic with respect to a Fuchsian group Γ. If*

$$\iint_F (1-|z|^2)^{p-2}\left\{\frac{|\mathrm{grad}\,h(z)|}{1+h^2(z)}\right\}^p dx\,dy < \infty,$$

where F is a fundamental region of Γ, then h is a strongly normal function with respect to Γ, and consequently, f is a strongly normal function with respect to Γ.

The proof of Theorem 6 is almost the same as that of Theorem 5. First, we may assume that $\bar{F}$ is not a compact subset of D and, for a fixed positive number $\sigma < 1$, we have positive numbers r_1, r_2 and r_3 such that

$$\iint_{F\setminus D_{r_1}} (1-|z|^2)^{p-2}\left\{\frac{|\mathrm{grad}\,h(z)|}{1+h^2(z)}\right\}^p dx\,dy \le \sigma\pi,$$

$r_1 < r_2 < 1$, $d(r_1, r_2) = 1$, and no point in $F \setminus D_{r_3}$ is equivalent to a point in D_{r_2}. For $z' \in F \setminus D_{r_3}$, set $f_1(z) = f(z) - i\tilde{h}(z')$ and $\Delta(z') = \{z \in D : d(z, z') < 1\}$. We have E' and E such that $E' \subset \Delta(z')$, $f_1\big(\Delta(z')\big) = f_1(E')$, and $E \subset F \setminus D_{r_1}$ is equivalent to E'. Denoting by A the spherical area of $f_1\big(\Delta(z')\big) = f_1(E')$, without consideration of multiplicity, we have

$$\begin{aligned}
A &\le \iint_{E'} \{f_1^{\#}(z)\}^2\,dx\,dy \\
&\le \left\{\iint_{F\setminus D_{r_1}} (1-|z|^2)^{p-2}\{f_1^{\#}(z)\}^p\,dx\,dy\right\}^{2/p} \left\{\iint_{\Delta(z')} \frac{dx\,dy}{(1-|z|^2)^2}\right\}^{1-2/p} \\
&\le \left\{\iint_{F\setminus D_{r_1}} (1-|z|^2)^{p-2}\left\{\frac{|\operatorname{grad} h(z)|}{1+h^2(z)}\right\}^p dx\,dy\right\}^{2/p} \left\{\iint_{\Delta(z')} \frac{dx\,dy}{(1-|z|^2)^2}\right\}^{1-2/p} \\
&\le (\sigma\pi)^{2/p} C^{1-2/p}.
\end{aligned}$$

Now, just as in the proof of Theorem 5, we have

$$(1-|z'|^2)f_1^{\#}(z') \le \frac{1}{r}\left\{\frac{\sigma^{2/p}(C/\pi)^{1-2/p}}{1-\sigma^{2/p}(C/\pi)^{1-2/p}}\right\}^{1/2}.$$

Note that, since

$$f_1^{\#}(z') = \frac{|\operatorname{grad} h(z')|}{1+h^2(z')},$$

we have

$$(1-|z'|^2)\frac{|\operatorname{grad} h(z')|}{1+h^2(z')} \le \frac{1}{r}\left\{\frac{\sigma^{2/p}(C/\pi)^{1-2/p}}{1-\sigma^{2/p}(C/\pi)^{1-2/p}}\right\}^{1/2}.$$

Letting $\sigma \longrightarrow 0$ completes the proof of Theorem 6.

4. **Remarks.** (1) Theorem 3 is a special case of Theorem 5. In fact, Theorem 3 is also a special case of a theorem proved by Aulaskari, Hayman, and Lappan[1], which states that if f is a rotation automorphic function with respect to a finitely generated Fuchsian group Γ and

$$\iint_F (1-|z|^2)^{p-2}\{f^{\#}(z)\}^p\,dx\,dy < \infty$$

for some $p \ge 2$, where F is a fundamental region of Γ, then f is a normal function and also

$$\lim_{|z|\longrightarrow 1, z\in F} (1-|z|^2)f^{\#}(z) = 0.$$

(2) The examples $f_n(z) = (z-1)^{1/n}$, for $n = 1, 2, \cdots$, show that the conclusion of Theorem 3 is sharp in the sense that $(1-|z|^2)f^{\#}(z)$ cannot be replaced by $(1-|z|^2)^{1-\epsilon}f^{\#}(z)$ with an arbitrarily small $\epsilon > 0$. In fact, the functions $f_n(z)$ are all bounded univalent functions, so they have images of finite spherical area. On the other hand, a simple calculation gives, for $0 < x < 1$,

$$(1-x^2)^{1-\epsilon} f_n^{\#}(x) \geq \frac{1}{5n}(1-x)^{1/n-\epsilon}.$$

For a given ϵ, taking n so large that $1/n < \epsilon$, we have

$$(1-x^2)^{1-\epsilon} f_n^{\#}(x) \longrightarrow \infty, \quad x \longrightarrow 1.$$

(3) The following example shows that the converse of Theorem 3 is not true. Consider the functions

$$g(z) = \exp\left(\frac{z+1}{z-1}\right), \quad f(z) = (z-1)^{1/2} g(z).$$

We have

$$g'(z) = -\frac{2g(z)}{(z-1)^2},$$

$$f'(z) = (z-1)^{1/2} g'(z) + \frac{1}{2}(z-1)^{-1/2} g(z) = \frac{1}{2}(z-5)(z-1)^{-3/2} g(z).$$

Thus, for $z \in D$,

$$f^{\#}(z) \leq |f'(z)| = \frac{3}{|z-1|^{3/2}} \exp\left(-\frac{1-|z|^2}{|z-1|^2}\right)$$

$$(1-|z|^2)^{3/4} f^{\#}(z) \leq 3\left(\frac{1-|z|^2}{|z-1|^2}\right)^{3/4} \exp\left(-\frac{1-|z|^2}{|z-1|^2}\right).$$

Hence, letting

$$M = \max_{r>0} r^{3/4} e^{-r},$$

we have

$$(1-|z|^2)^{3/4} f^{\#}(z) \leq 3M$$

for $z \in D$.

On the other hand, $f(z)$ has an image of infinite spherical area. To show this, we go to the left half plane $L = \{\zeta \in \mathbb{C} : \Re\zeta < 0\}$. Set

$$\zeta = \frac{z+1}{z-1}.$$

Then

$$z - 1 = \frac{2}{\zeta - 1}, \quad f(z) = \phi(\zeta) = \frac{2^{1/2} e^{\zeta}}{(\zeta - 1)^{1/2}}.$$

What we want to prove is that $\phi(\zeta)$ maps L onto a covering surface with infinite spherical area, that is,

$$A = \iint_L \{\phi^{\#}(\zeta)\}^2 \, d\xi \, d\eta = \infty.$$

For $\zeta \in L$, we have

$$\phi'(\zeta) = \frac{e^\zeta}{2^{1/2}(\zeta-1)^{1/2}}\left(2 - \frac{1}{\zeta-1}\right),$$

$$|\phi'(\zeta)| \geq \frac{|e^\zeta|}{2^{1/2}|\zeta-1|^{1/2}}, \quad |\phi^{\#}(\zeta)| \geq \frac{|e^\zeta|}{5|\zeta-1|^{1/2}}.$$

Let $\zeta = \xi + i\eta$ and let

$$G_n = \{\zeta \in \mathbb{C} : -1 < \xi < 0, 2n\pi < \eta < 2(n+1)\pi\}$$

for $n = 0, 1, \cdots$. Then,

$$\begin{aligned} A &\geq \sum_{n=0}^{\infty} \iint_{G_n} \{\phi^{\#}(\zeta)\}^2 \, d\xi \, d\eta \geq \frac{1}{25} \sum_{n=0}^{\infty} \iint_{G_n} \frac{|e^\zeta|^2}{|\zeta-1|} \, d\xi \, d\eta \\ &\geq \frac{1}{50} \sum_{n=0}^{\infty} \frac{1}{\sqrt{1+(n+1)^2\pi^2}} \iint_{G_n} |e^\zeta|^2 \, d\xi \, d\eta \\ &= \frac{1}{50}\pi(1-e^{-2}) \sum_{n=0}^{\infty} \frac{1}{\sqrt{1+(n+1)^2\pi^2}} = \infty. \end{aligned}$$

(4) We cannot omit the assumption that $\tilde{h}$ is also automorphic with respect to Γ in Theorem 6. Indeed, Aulaskari and Lappan [2] constructed a holomorphic function $W(z)$ with the following properties:

(1) the real part $h(z)$ is automorphic with respect to a Fuchsian group Γ;
(2) $\iint_F |W'(z)|^2 \, dx \, dy < \infty$, where F is a fundamental region of Γ;
(3) $W(z)$ and consequently $h(z)$ is not normal.

NOTE. Professor Joel Schiff, after receiving a preliminary version of the present paper, wrote us that it was proved on pages 39 and 180 of his book [9], that Dirichlet-finite functions on D are normal and that it would appear from the proof that such functions are actually strongly normal, and in fact, strongly Bloch.

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Canad. Math. Bull. Vol. 39 (4), 1996 pp. 420–428

THE FAILURE OF APPROXIMATE INNER CONJUGACY FOR STANDARD DIAGONALS IN REGULAR LIMIT ALGEBRAS

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ABSTRACT. AF C*-algebras contain natural AF masas which, here, we call *standard diagonals*. Standard diagonals are unique, in the sense that two standard diagonals in an AF C*-algebra are conjugate by an approximately inner automorphism. We show that this uniqueness fails for non-selfadjoint AF operator algebras. Precisely, we construct two standard diagonals in a particular non-selfadjoint AF operator algebra which are not conjugate by an approximately inner automorphism of the non-selfadjoint algebra.

1. **Introduction.** Approximately finite-dimensional (AF) C*-algebras contain natural AF maximal abelian self-adjoint subalgebras (masas) which, here, we call *standard diagonals*. A standard diagonal arises from a nested matrix unit system for a dense finite-dimensional subalgebra chain and it serves a useful role for coordinatisation, classification and representation theory; see [6] for example. Standard diagonals are regular in the sense, borrowed from von Neumann algebra theory, that the set of partial isometries which normalise the diagonal generates the containing algebra. Indeed, all matrix units normalise the diagonal. In the non-selfadjoint setting it is the so-called *regular* limit algebras that contain (regular) standard diagonals. On the other hand there are limit algebras that are not regular [3].

Just as in the (trivial) finite-dimensional case, any two standard diagonals in an AF C*-algebra are conjugate by an approximately inner automorphism. In other words, there is an automorphism α mapping one standard diagonal onto the other and a sequence of unitaries $\{u_k\}_{k\in\mathbb{N}}$ in the AF C*-algebra so that α has the representation

$$\alpha(a) = \lim_{k\to\infty} u_k a u_k^*.$$

This uniqueness is implicit in the discussions of the uniqueness of AF groupoids given by Renault [5] and Kreiger [2], and the fact that automorphisms of AF C*-algebras that fix K_0 are necessarily approximately inner. For a direct proof, see Theorem 5.7 of [4].

We show that, unexpectedly, this natural uniqueness can fail for non-selfadjoint approximately finite operator algebras and even in the purely algebraic setting, for regular algebraic direct limits of digraph algebras. We do this by constructing two standard diagonals in a particular non-selfadjoint AF operator algebra for which this uniqueness fails. However, there is an automorphism (*not* approximately inner) that maps one standard diagonal to the other. Hence our example leaves the following question open: for

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The first author was partly supported by an NSERC of Canada Postdoctoral Fellowship.

Received by the editors May 17, 1995.

AMS subject classification: Primary: 47D25; Secondary: 46K50.

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two standard diagonals in a non-selfadjoint AF operator algebra, is there always a star-extendible automorphism that maps one to the other?

We now give a precise formulation of the main result.

Let $\{B_k, \beta_k\}_{k=1}^{\infty}$ be a direct system of finite-dimensional C*-algebras B_k and C*-algebra homomorphisms $\beta_k: B_k \longrightarrow B_{k+1}$. There are many ways to choose matrix unit systems for $B_1, B_2, \ldots$ so that each β_k maps matrix units to sums of matrix units. Each such choice gives rise to an abelian algebra D_0 in the algebraic direct limit $B_0 = \operatorname{alg} \varinjlim B_k$, namely $D_0 = \operatorname{alg} \varinjlim D_k$, where D_k is the subalgebra spanned by the diagonal matrix units. Since matrix units are mapped to sums of matrix units by each β_k, it follows that these matrix units normalise D_0 and so D_0 is regular in B_0. We refer to D_0 as a *standard diagonal* of B_0. Standard diagonals, even when arising from different direct systems for B_0, are conjugate by an approximately inner star automorphism of B_0. The same is also true for standard diagonals $D = \varinjlim D_k$ in the AF C*-algebra $B = \varinjlim B_k$.

Consider the intermediate algebra A_0 with $D_0 \subseteq A_0 \subseteq B_0$ given by $A_0 = \operatorname{alg} \varinjlim A_k$ where the A_k are algebras, which are not necessarily self-adjoint, such that, for all k,

$$D_k \subseteq A_k \subseteq B_k \text{ and } \beta_k(A_k) \subseteq A_{k+1}.$$

The finite-dimensional algebras A_k are known variously as poset algebras, incidence algebras, CSL algebras and digraph algebras. We adopt the digraph terminology, which is motivated by the observation that if B_k has matrix unit system $\left\{e_{ij}^{(k)} \mid (i,j) \in I_k\right\}$ then the matrix units $e_{ij}^{(k)}$ in A_k are labelled by the edges (j,i) of a reflexive transitive digraph. A subalgebra E_0 of the intermediate algebra A_0 is called a *standard diagonal* if there exists a presentation $A_0 = \operatorname{alg} \varinjlim A_k$, as above, so that $E_0 = D_0$. The following nonconjugacy result contrasts with both the self-adjoint context and the finite-dimensional case.

THEOREM 1. *There exists a regular limit algebra* $A_0 = \operatorname{alg} \varinjlim A_k$ *which contains standard diagonals* C_0 *and* C_0' *which are not conjugate by an approximately inner automorphism of* A_0*. Moreover the norm closures of* C_0 *and* C_0' *in the operator algebra* $A = \varinjlim A_k$ *are standard diagonals which are not conjugate by an approximately inner automorphism of* A*.*

The key to the proof is the construction, in finite dimensions, of homomorphisms of digraph algebras, $\alpha: A_1 \longrightarrow A_2$ and $\beta: A_2 \longrightarrow A_3$, so that neither α nor β map matrix units to sums of matrix units (for any choices of matrix units), yet the composition $\beta \circ \alpha$ is nondegenerate and does map matrix units to sums of matrix units.

Relevant theory for non-selfadjoint limit algebras is given in [4]. Nevertheless, for completeness we give the needed terminology and basic facts in the next section.

2. **Terminology and Preliminaries.** The limit algebras A_0, A may be defined, ab initio, without reference to the containing C*-algebras B_0, B, by means of *regular* homomorphisms, the most natural morphisms for digraph algebras.

DEFINITION 2. Let $A_1 \subseteq M_{n_1}$ and $A_2 \subseteq M_{n_2}$ be digraph algebras. An algebra homomorphism $\varphi: A_1 \longrightarrow A_2$ is *star-extendible* if there exists a C*-algebra homomorphism $\tilde{\phi}: C^*(A_1) \longrightarrow C^*(A_2)$ for which $\varphi = \tilde{\phi}_{|A_1}$.

Given a star-extendible homomorphism $\varphi: A_1 \longrightarrow A_2$ it may not be possible to choose matrix units for A_1 (and M_{n_1}) and for A_2 (and M_{n_2}) so that φ maps matrix units to sums of matrix units. The homomorphisms for which this is so are the star-extendible *regular* homomorphisms.

DEFINITION 3. A star-extendible digraph algebra homomorphism $\varphi: A_1 \longrightarrow A_2$ is of *multiplicity one* if $\operatorname{rank}\big(\varphi(a)\big) \leq \operatorname{rank}(a)$ for all a in A_1, and is *regular* if it is a direct sum of multiplicity one star-extendible homomorphisms.

(It is convenient to allow the zero map to have multiplicity one.) The rank condition need only be specified for rank one projections in A_1. In general, regularity for a digraph algebra homomorphism is defined in terms of normalisers. However, for star-extendible maps, this is equivalent to the definition above; see Exercise 4.1 of [4].

The restriction morphisms $\beta_k: A_k \longrightarrow A_{k+1}$ given in the introduction are easily seen to be regular. But conversely, if $\{A_k, \varphi_k\}$ is a direct system of digraph algebras with regular star-extendible homomorphisms $\varphi_k: A_k \longrightarrow A_{k+1}$, then it is possible to successively choose matrix unit systems $\left\{e_{ij}^{(k)} : (i,j) \in I_k\right\}$ for each $B_k = C^*(A_k)$ so that each A_k is spanned by $\left\{e_{ij}^{(k)} : e_{ij}^{(k)} \in A_k\right\}$ and each φ_k maps matrix units to sums of matrix units.

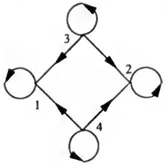
DEFINITION 4. Let C be a self-adjoint subalgebra of a C*-subalgebra A. Then the (partial isometry) *normaliser* of C in A is the set $N_C(A)$ of partial isometries v in A for which vcv^* and v^*cv belong to C for all c in C.

The normaliser of a standard diagonal in a digraph algebra coincides with the set of partial isometries v which are orthogonal sums of unimodular multiples of matrix units. As mentioned above, a star-extendible homomorphism $\varphi: A_1 \longrightarrow A_2$ is regular if and only if there exist diagonals (masas) $C_1 \subseteq A_1$ and $C_2 \subseteq A_2$ such that $\varphi\big(N_{C_1}(A_1)\big) \subseteq N_{C_2}(A_2)$. Furthermore for each diagonal C_1 chosen arbitrarily in A_1 there exists a diagonal C_2 in A_2 for which the inclusion holds.

Repeating this observation, it follows that if $\{A_k, \alpha_k\}$ is a regular direct system of digraph algebras, and if $C_k \subseteq A_k$ are chosen masas (or are given *a priori* as in the introduction) with the normalising property $\alpha_k\big(N_{C_k}(A_k)\big) \subseteq N_{C_{k+1}}(A_{k+1})$ for all k, then the abelian algebra $C_0 = \operatorname{alg} \varinjlim C_k$, which is a masa in A_0, has nontrivial normaliser $N_{C_0}(A_0)$ in $A_0 = \operatorname{alg} \varinjlim A_k$, and in fact A_0 is generated as an algebra by $N_{C_0}(A_0)$. Similar assertions hold in the case of Banach algebra limits. To summarise, standard diagonals are regular in the sense mentioned in the introduction.

Let D_4 be the 4-cycle digraph and $A(D_4) \subseteq M_4(\mathbb{C})$ be the digraph algebra spanned by the standard matrix units e_{ij} where (j, i) is an edge of D_4, *i.e.*,

D_4 is



$$A(D_4) = \begin{bmatrix} * & 0 & * & * \\ 0 & * & * & * \\ 0 & 0 & * & 0 \\ 0 & 0 & 0 & * \end{bmatrix}$$

The limit algebras of Theorem 1 are based on the building blocks $A(D_4) \otimes M_n(\mathbb{C})$.

There are four symmetries of D_4: the identity, the reflection that exchanges 1 with 2, the rotation that exchanges 1 with 2 and 3 with 4, and the reflection that exchanges 3 with 4. They induce four automorphisms of $A(D_4)$, which we label, respectively, θ_1, θ_2, θ_3 and θ_4. As in [4] we say that a homomorphism $\alpha: A(D_4) \otimes M_n \longrightarrow A(D_4) \otimes M_m$ is *rigid* if it is a direct sum of multiplicity one homomorphisms, each of which is inner conjugate to an embedding γ where $\gamma(a \otimes b) = \theta(a) \otimes (b \oplus 1_{m-n})$ and θ is one of the θ_i. Thus a rigid embedding is a regular embedding without degenerate summands.

Let us say that an ordered set $\{v_1, v_2, v_3, v_4\}$ of rank one partial isometries in $A(D_4) \otimes M_n$ is an *essential 4-cycle* if $v_4^* v_3 v_2^* v_1 = v_1^* v_1$ and each v_i belongs to one of the spaces $e_{13} \otimes M_n, e_{14} \otimes M_n, e_{24} \otimes M_n, e_{23} \otimes M_n$ with no two belonging to the same such space. Then rigid embeddings are characterised by the fact that they map an essential 4-cycle to a 4-cycle of partial isometries which splits into a direct sum of essential 4-cycles.

The significance of rigid embeddings is that they generate a wide class of non-self-adjoint limit algebras (see [4] and [1]) and they are determined by their induced map on $K_0 \oplus H_1$. Here $H_1\big(A(D_4) \otimes M_n\big)$ can be taken to be the integral simplicial homology of the simplical complex of the undirected graph $\bar{D}_4 \times K_n$, where K_n is the complete n-vertex graph and $\bar{D}_4$ is the undirected 4-cycle graph. Thus, $H_1\big(A(D_4) \otimes M_n\big) = \mathbb{Z}$ and, with consistent identifications of $H_1\big(A(D_4) \otimes M_n\big)$ and $H_1\big(A(D_4) \otimes M_m\big)$, $H_1\gamma = [\pm 1]$. Precisely, $H_1\gamma$ is the 1×1 matrix $[+1]$ if θ is the identity θ_1 or the rotation θ_3 and is $[-1]$ if θ is one of the two reflections.

3. **Irregular morphisms and nonconjugate diagonals.** Let $A_k = A(D_4) \otimes M_{2^{k-1}}$, for $k = 1, 2, \ldots$, and view A_k as the subalgebra of $M_4 \otimes M_{2^{k-1}}$ consisting of block matrices

$$a = \begin{bmatrix} A & & X & Y \\ & B & W & Z \\ & & C & \\ & & & D \end{bmatrix}$$

where the unspecified entries are zero. The cyclical labeling X, Y, Z, W is useful, as we shall see, for keeping track of the homology of images of an essential 4-cycle of matrix units in A_k. Define $\alpha_k \colon A_k \longrightarrow A_{k+1}$ by setting

$$\alpha_k(a) = \left[\begin{array}{cccccccc}
A & 0 & & & X' & Y' & X' & Y' \\
0 & B & & & W' & -Z' & -W' & Z' \\
 & & A & 0 & X' & -Y' & -X' & Y' \\
 & & 0 & B & W' & Z' & W' & Z' \\
 & & & & C & 0 & & \\
 & & & & 0 & D & & \\
 & & & & & & C & 0 \\
 & & & & & & 0 & D
\end{array}\right]$$

where $X' = \frac{1}{\sqrt{2}}X, Y' = \frac{1}{\sqrt{2}}Y, Z' = \frac{1}{\sqrt{2}}Z, W' = \frac{1}{\sqrt{2}}W$.

LEMMA 5. *For each k, α_k is a star-extendible algebra injection of multiplicity 2 which is irregular. However, the compositions $\alpha_{k+1} \circ \alpha_k$ are rigid and $H_1(\alpha_{k+1} \circ \alpha_k) = 0$.*

A composition of irregular embeddings may be regular for rather trivial or degenerate reasons. For instance, the composition may map into the self-adjoint subalgebra of the final algebra. The importance of the α_k is that their compositions are rigid and so do not have this kind of degeneracy.

PROOF. It is easy to check that each α_k is a star-extendible algebra injection of multiplicity 2. To see that α_1 is irregular, note that $\alpha_1(e_{1,3})$ is not inner conjugate to the sum of two rank one partial isometries in A_2 whose initial and final projections belong to A_2, and yet this is a necessary condition for regularity. Similar remarks apply to α_k for $k \geq 1$.

It remains only to show that $\alpha_{k+1} \circ \alpha_k$ is rigid. To see this observe first that $\alpha_{k+1} \circ \alpha_k(a)$ is the block matrix

$$\left[\begin{array}{cccccccccccccccc}
A & 0 & 0 & 0 & & & & & x & y & x & y & x & y & x & y \\
0 & B & 0 & 0 & & & & & w & -z & -w & z & w & -z & -w & z \\
0 & 0 & A & 0 & & & & & x & -y & x & -y & -x & y & -x & y \\
0 & 0 & 0 & B & & & & & w & z & -w & -z & -w & -z & w & z \\
 & & & & A & 0 & 0 & 0 & x & y & -x & -y & -x & -y & x & y \\
 & & & & 0 & B & 0 & 0 & w & -z & w & -z & -w & z & -w & z \\
 & & & & 0 & 0 & A & 0 & x & -y & -x & y & x & -y & -x & y \\
 & & & & 0 & 0 & 0 & B & w & z & w & z & w & z & w & z \\
 & & & & & & & & C & 0 & 0 & 0 & & & & \\
 & & & & & & & & 0 & D & 0 & 0 & & & & \\
 & & & & & & & & 0 & 0 & C & 0 & & & & \\
 & & & & & & & & 0 & 0 & 0 & D & & & & \\
 & & & & & & & & & & & & C & 0 & 0 & 0 \\
 & & & & & & & & & & & & 0 & D & 0 & 0 \\
 & & & & & & & & & & & & 0 & 0 & C & 0 \\
 & & & & & & & & & & & & 0 & 0 & 0 & D
\end{array}\right]$$

where $x = \frac{1}{2}X, y = \frac{1}{2}Y, z = \frac{1}{2}Z$ and $w = \frac{1}{2}W$. In particular, the image of each of the matrix units $e_{13}, e_{14}, e_{24}, e_{23}$ has the block form

$$\begin{bmatrix} 0 & & V_1 & V_2 \\ & 0 & V_4 & V_3 \\ & & 0 & \\ & & & 0 \end{bmatrix}$$

where *each V_i is a partial isometry*. It is this fortuitous property which is crucial for the construction.

If $u \in A_{k+2}$ is the block diagonal unitary whose four diagonal entries are each

$$\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix},$$

then $u\big(\alpha_{k+1} \circ \alpha_k(a)\big)u^*$ has the form

$$\left[\begin{array}{cccccccccccccccc}
A & 0 & 0 & 0 & & & & & X & 0 & 0 & 0 & 0 & Y & 0 & 0 \\
0 & B & 0 & 0 & & & & & 0 & 0 & -W & 0 & 0 & 0 & 0 & Z \\
0 & 0 & A & 0 & & & & & 0 & -Y & 0 & 0 & -X & 0 & 0 & 0 \\
0 & 0 & 0 & B & & & & & 0 & 0 & 0 & -Z & 0 & 0 & W & 0 \\
& & & & A & 0 & 0 & 0 & 0 & 0 & -X & 0 & 0 & 0 & 0 & Y \\
& & & & 0 & B & 0 & 0 & W & 0 & 0 & 0 & 0 & Z & 0 & 0 \\
& & & & 0 & 0 & A & 0 & 0 & 0 & 0 & Y & 0 & 0 & -X & 0 \\
& & & & 0 & 0 & 0 & B & 0 & Z & 0 & 0 & W & 0 & 0 & 0 \\
& & & & & & & & C & 0 & 0 & 0 & & & & \\
& & & & & & & & 0 & D & 0 & 0 & & & & \\
& & & & & & & & 0 & 0 & C & 0 & & & & \\
& & & & & & & & 0 & 0 & 0 & D & & & & \\
& & & & & & & & & & & & C & 0 & 0 & 0 \\
& & & & & & & & & & & & 0 & D & 0 & 0 \\
& & & & & & & & & & & & 0 & 0 & C & 0 \\
& & & & & & & & & & & & 0 & 0 & 0 & D
\end{array}\right]$$

The minus signs are easily removed, by a further diagonal conjugation, and it follows that $\alpha_{k+1} \circ \alpha_k$ is a regular embedding. Furthermore $\alpha_{k+1} \circ \alpha_k$ is rigid, since the image of an essential 4-cycle can be seen to be the direct sum of 4 essential 4-cycles, and inspecting the orientation of these cycles shows that $H_1(\alpha_{k+1} \circ \alpha_k) = 0$. ■

Consider now the two regular direct systems

$$\mathcal{S}_{\text{even}} = \{A_{2k}, \alpha_{2k+1} \circ \alpha_{2k}\}, \quad \mathcal{S}_{\text{odd}} = \{A_{2k-1}, \alpha_{2k} \circ \alpha_{2k-1}\}.$$

Note that they determine the same limit algebra, call it $\mathcal{A}$, and yet the identity isomorphism $\mathcal{A} \longrightarrow \mathcal{A}$ is not induced by a regular isomorphism between $\mathcal{S}_{\text{even}}$ and $\mathcal{S}_{\text{odd}}$. By a *regular* isomorphism, we mean one for which there is a commmuting diagram between $\mathcal{S}_{\text{even}}$ and $\mathcal{S}_{\text{odd}}$ in which all the interlacing maps are regular. However in this case the connecting maps of such a commuting diagram must be compositions of an odd number of consecutive α_k and these will be irregular. The next theorem shows that we can construct the required counterexamples in $\mathcal{A}$, proving Theorem 1.

THEOREM 6. *The limit algebra* $\mathcal{A} = \varinjlim\big(A(D_4)\otimes M_{2^{k-1}}, \alpha_k\big)$ *contains two standard diagonals which are not conjugate by an approximately inner automorphism of* $\mathcal{A}$.

PROOF. Recall $A_k = A(D_4)\otimes M_{2^{k-1}}$. Choose masas $C_{2k} \subseteq A_{2k}$ successively so that the normaliser of C_{2k} contains the image under $\alpha_{2k-1}\circ\alpha_{2k-2}$ of the normaliser of C_{2k-2}. Similarly choose $C_1, C_3, \ldots$. Then $C = \varinjlim C_{2k}$ and $C' = \varinjlim C_{2k+1}$ are standard diagonals in $\mathcal{A}$.

For notational convenience we can henceforth regard the given maps α_k as inclusion maps.

Suppose, by way of contradiction, that $\gamma\colon \mathcal{A} \longrightarrow \mathcal{A}$ is an automorphism so that

$$\gamma(C) = C' \text{ and } \gamma(a) = \lim_{k\to\infty} u_k a u_k^*,$$

for some sequence of unitaries $u_k \in \mathcal{A}$. If a unitary in $\mathcal{A}$ is close to an element of the subalgebra $A(D_4)\otimes M_{2^{k-1}}$, then it is close to a unitary in $A(D_4)\otimes M_{2^{k-1}}$. For this reason we may assume that $u_k \in A_k$ for all k.

Since γ is isometric it follows that γ maps the normaliser of C in $\mathcal{A}$ into the normaliser of C' in $\mathcal{A}$ (see Proposition 7.1 of [4]). In view of the structure of normalising partial isometries (*i.e.*, Lemma 5.5 of [4]), there exists m_1 such that

$$\gamma(A_1) \subseteq \tilde{A}_{2m_1}$$

where $\tilde{A}_{2m_1}$ is the algebra generated by A_{2m_1} and C'. Increasing m_1 if necessary, we can express the resulting restriction map $\gamma_r\colon A_1 \longrightarrow \tilde{A}_{2m_1}$, as a composition $(\operatorname{Ad} c_1)\circ\gamma_1$ where $\gamma_1\colon A_1 \longrightarrow A_{2m_1}$ is regular, mapping C_1 into C_{2m_1}, and c_1 is a unitary element of C. (This follows immediately from the elementary C*-algebraic fact that the extension $\tilde{\gamma}_r\colon \mathrm{C}^*(A_1) \longrightarrow \mathrm{C}^*(\tilde{A}_{2m_1})$ has such a factored form.) Increasing m_1 again, if necessary, γ_r is close to $(\operatorname{Ad} c_2)\circ\gamma_1$ for some unitary element c_2 of A_{2m_1}. But by hypothesis, we may also assume that γ_r is close to $\operatorname{Ad} u$ with $u = u_{2m_1}$ a unitary in A_{2m_1}. Thus the inclusion map $i\colon A_1 \longrightarrow A_{2m_1}$ is close to the regular map $(\operatorname{Ad} u^*)\circ(\operatorname{Ad} c_2)\circ\gamma_1$. In particular viewing $i(e_{13})$ as a 4×4 block matrix in A_{2m_1} we deduce that the $(1,3)$ block is close to a partial isometry, and in particular has norm close to one.

On the other hand, $i = \beta\circ\alpha_1$, where β, being an even composition of the given maps α_k, is a rigid embedding. Since β is a direct sum of multiplicity one rigid embeddings it becomes clear that the norm of the $(1,3)$ block of $i(e_{13})$ equals the norm of the $(1,3)$ block of $\alpha_1(e_{13})$, *i.e.*, $1/\sqrt{2}$. This contradiction completes the proof. ∎

REMARKS. 1. This theorem has some purely C*-algebraic consequences for extensions of approximately inner automorphisms.

Let $\mathcal{E} = \mathcal{A}\cap\mathcal{A}^*$ and let $\mathcal{F} = \mathrm{C}^*(\mathcal{A})$, both AF C*-algebras. Then there is an approximately inner automorphism of $\mathcal{E}$, say ϕ, that carries C to C'. However, if $\{u_k\}$ is any sequence of unitaries in $\mathcal{E}$ satisfying $\phi(x) = \lim_k u_k x u_k^*$ for all $x \in \mathcal{E}$, then this limit representation does not extend to $\mathcal{F}$. That is, there must be some $y \in \mathcal{F}$ so that $\lim_k u_k y u_k^*$ does not exist. Otherwise, we would have an approximately inner automorphism of $\mathcal{F}$

that restricts to an approximately inner automorphism of $\mathcal{A}$ (as $u_k, u_k^* \in \mathcal{A}$), contradicting Theorem 6.

If we do not require that both C and C' are regular in $\mathcal{F}$, then it is possible to construct standard diagonals C, C' in the AF C*-algebra $\mathcal{E}$ so that there is an approximately inner automorphism of $\mathcal{E}$ carrying C to C' but no such automorphism of $\mathcal{F}$. Obviously, the automorphism of $\mathcal{E}$ does not extend to $\mathcal{A}$ in this case. The automorphism of $\mathcal{E}$ in the setting of Theorem 6 fails to extend for different reasons, as C and C' are regular in $\mathcal{F}$ and so there is an approximately inner automorphism of $\mathcal{F}$ that carries C to C'. By the previous paragraph, the implementing unitaries of this automorphism of $\mathcal{F}$ cannot be chosen in $\mathcal{E}$.

2. We give another direct system for $\mathcal{A}$, showing $\mathcal{A}$ can be given by quite conventional regular embeddings. Let $\beta_k: A(D_4) \otimes M_{4^{k-1}} \longrightarrow A(D_4) \otimes M_{4^k}$ be regular embeddings for which

$$\begin{aligned}\beta_k(a \otimes b) = \theta_1(a) \otimes (b \oplus 0 \oplus 0 \oplus 0) + \theta_2(a) \otimes (0 \oplus b \oplus 0 \oplus 0)\\ + \theta_3(a) \otimes (0 \oplus 0 \oplus b \oplus 0) + \theta_4(a) \otimes (0 \oplus 0 \oplus 0 \oplus b),\end{aligned}$$

where $\theta_1, \dots, \theta_4$ are the permutation automorphisms of $A(D_4)$. The embeddings β_k are in fact rigid embeddings with

$$K_0\beta_k = \begin{bmatrix} 2 & 2 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 2 & 2 \end{bmatrix}, \quad H_1\beta_k = [0].$$

Hence $\alpha_{2k+1} \circ \alpha_{2k}$ is inner conjugate to β_k, for all k. With the notation above, $\mathcal{A}$ is isomorphic to $\varinjlim\big(A(D_4) \otimes M_{4^{k-1}}, \beta_k\big)$.

3. The diagonals C, C' above are automorphically conjugate, by which we mean that there is a star-extendible automorphism of $\mathcal{A}$ which maps C to C'. To see this one can appeal to Theorem 11.21 of [4] or, much more simply, argue as follows.

Recall that θ_1 is the identity automorphism of $A(D_4)$ and θ_2 and θ_4 are the two reflections. Let $\eta_1: A_1 \longrightarrow A_2$ be the standard regular embedding from $A(D_4) \otimes \mathbb{C}$ to $A(D_4) \otimes M_2$ with

$$\eta_1(a \otimes b) = \theta_1(a) \otimes \begin{bmatrix} b & 0 \\ 0 & 0 \end{bmatrix} + \theta_2(a) \otimes \begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix}.$$

Replacing η_1 by an inner conjugate we can arrange $\eta_1(C_1) \subseteq C_2$. Let $\eta_2: A_2 \longrightarrow A_3$ be the standard regular embedding from $A(D_4) \otimes M_2$ to $A(D_4) \otimes M_4$ with

$$\eta_2(a \otimes b) = \theta_1(a) \otimes \begin{bmatrix} b & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \end{bmatrix} + \theta_4(a) \otimes \begin{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & b \end{bmatrix}.$$

Now $\eta_2 \circ \eta_1$ is rigid, and furthermore has the same $K_0 \oplus H_1$ data as β_1. We may therefore replace η_2 by an inner conjugate embedding so that $\eta_2 \circ \eta_1 = \beta_1$. Continue in this way (with η_3 defined in terms of θ_1 and θ_2, η_4 in terms of θ_1 and θ_4, and so on) to obtain an automorphism $\lim \eta_{2k-1}$ from the pair $(\mathcal{A}, C)$ to the pair $(\mathcal{A}, C')$.

It is an important open problem whether standard diagonals are automorphically conjugate in general. A positive answer would show, for example, that homology groups considered in [4, Chapter 11] and [1] are invariants for the non-selfadjoint AF operator algebra as well as for the pair of algebra and standard diagonal.

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Canad. Math. Bull. Vol. 39 (4), 1996 pp. 429–437

C^*-ALGEBRAS OF REAL RANK ZERO WHOSE K_0'S ARE NOT RIESZ GROUPS

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ABSTRACT. Examples are constructed of stably finite, unital, separable C^*-algebras A of real rank zero such that the partially ordered abelian groups $K_0(A)$ do not satisfy the Riesz decomposition property. This contrasts with the result of Zhang that projections in C^*-algebras of real rank zero satisfy Riesz decomposition. The construction method also produces a stably finite, unital, separable C^*-algebra of real rank zero which has the same K-theory as an approximately finite dimensional C^*-algebra, but is not itself approximately finite dimensional.

1. **Introduction.** The Riesz decomposition property has long been an important tool in the K-theory of C^*-algebras. We recall the definition of this property for a partially ordered abelian group G: whenever $x, y_1, y_2 \in G^+$ with $x \leq y_1 + y_2$, there is a decomposition $x = x_1 + x_2$ with $x_i \in G^+$ such that $x_i \leq y_i$ for $i = 1, 2$. A directed, partially ordered abelian group satisfying Riesz decomposition is called a *Riesz group* for short. For example, Riesz decomposition was the key to the Effros-Handelman-Shen theorem [7] determining precisely which ordered groups appear as K_0-groups in Elliott's classification of AF (approximately finite dimensional) C^*-algebras [8]. It was also the fundamental property used by Handelman, Lawrence, and the author in building a structure theory for K_0 of finite Rickart C^*-algebras [15]. More recently, Riesz decomposition has played an important role in the analysis and classification of C^*-algebras of real rank zero. Zhang proved in [26, Corollary 1.3] that the projections in any C^*-algebra A of real rank zero satisfy Riesz decomposition with respect to Murray-von Neumann equivalence, that is, whenever p, q_1, q_2 are projections in A with $p \lesssim q_1 \oplus q_2$, there is an orthogonal decomposition $p = p_1 \oplus p_2$ for some projections $p_i \in A$ such that $p_i \lesssim q_i$ for $i = 1, 2$. This result was a key tool in his investigations of the structure of such algebras and their corona and multiplier algebras (*e.g.*, [26, 27]). In Elliott's recent classification of certain C^*-algebras of real rank zero [10], the Riesz decomposition property again made possible an axiomatic characterization of the range of the invariant: namely, every countable, weakly unperforated, graded Riesz group appears as $K_*(A) = K_0(A) \oplus K_1(A)$ for a separable, nuclear C^*-algebra A of real rank zero and stable rank one [9, Theorem 8.2].

If A is a C^*-algebra of real rank zero, Zhang's result implies that the monoid $V(A)$ of Murray-von Neumann equivalence classes of projections from $M_\infty(A)$ (*cf.* [1, 5.1.2]) has Riesz decomposition. In case the projections in A have cancellation (in particular, in case A has stable rank 1), it follows that $K_0(A)$ has Riesz decomposition. *A priori*,

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Received by the editors April 26, 1995.

AMS subject classification: 46L80, 19K14.

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however, the Riesz decomposition property can be lost in the passage from the monoid $V(A)$ to the ordered group $K_0(A)$. This phenomenon was exhibited in the context of von Neumann regular rings by Moncasi [19, Example], who constructed examples of stably finite, unital regular rings whose K_0's are not Riesz groups. We show that a C^*-analog of Moncasi's construction produces stably finite, unital C^*-algebras which have the same ordered K_0's as his examples. However, the algebras thus constructed do not have real rank zero. One final construction step is necessary to produce the examples announced in our title.

As an unexpected byproduct, our construction also produces an example of a stably finite, unital C^*-algebra of real rank zero which has the same K_0- and K_1-groups as an AF C^*-algebra, yet is not AF itself. This algebra is in a sense 'just outside' the class addressed by Elliott's classification – it is an extension of a Bunce-Deddens algebra by a direct sum of two copies of the algebra of compact operators, but it has stable rank 2 and so cannot be an inductive limit of direct sums of Elliott's building block algebras. In fact, it is not an approximately homogeneous algebra. The first example of such a 'bad' extension was constructed by Putnam (see [16, 2]). Our example may be an indication that the class of all stably finite, separable, nuclear C^*-algebras with real rank zero is too large to be classified by K-theory. In particular, stable finiteness may be too weak a finiteness condition (outside the class of simple C^*-algebras), and so for classification purposes a stronger condition should be imposed. One obvious candidate is stable rank one; another, suggested by Loring, is the 'extremal richness' property studied in [4]. However, S. Eilers and N. Larsen have put a dent in the latter suggestion by showing that the example constructed here is extremally rich [personal communication].

It should be emphasized that the question whether Riesz decomposition holds in K_0 is most natural in the setting of stably finite algebras, because for such algebras the natural pre-order relation on K_0 is a partial ordering [1, Proposition 6.3.3]. Failure of Riesz decomposition would be less surprising in a K_0-group which is not partially ordered.

For notation and definitions of unexplained terms, we refer the reader to [1] and [12]. Following a suggestion of the referee, we write $n \cdot e$ rather than $n. e$ for the orthogonal sum of n copies of a projection e (that is, the $n \times n$ diagonal matrix $\operatorname{diag}(e, e, \ldots, e)$).

We thank M. Dădarlăt and T. Loring for their comments on an earlier version of this paper.

2. Construction.

2.1. Fix a positive integer n, and let u_n denote the n-fold right shift on the Hilbert space ℓ^2, that is, the n-th power of the unilateral shift $(\alpha_1, \alpha_2, \ldots) \mapsto (0, \alpha_1, \alpha_2, \ldots)$. Then u_n is an isometry in $\mathcal{B}(\ell^2)$ and $1 - u_n u_n^*$ is a rank n projection. Set T_n equal to the (separable, unital) sub-C^*-algebra of $\mathcal{B}(\ell^2)$ generated by u_n together with the ideal $\mathcal{K}$ of compact operators on ℓ^2. In case $n = 1$, we have the standard Toeplitz algebra; as is well known, $T_1 / \mathcal{K} \cong C(\mathbb{T})$ by an isomorphism that links the coset $u_1 + \mathcal{K}$ with the standard unitary $z \in C(\mathbb{T})$, namely the inclusion map $z \colon \mathbb{T} \hookrightarrow \mathbb{C}$.

An alternative description of T_n can be obtained by adapting a construction used by Plastiras [21] and Bures [5]. Let $\Delta_n \colon T_1 \longrightarrow M_n(T_1)$ be the diagonal map, and set $T_n' =$

$\Delta_n(T_1) + M_n(\mathcal{K})$. It is clear that $T_n \cong T_n'$ by an isomorphism sending $\mathcal{K}$ to $M_n(\mathcal{K})$ and u_n to $\Delta_n(u_1)$. From this description we see that $T_n/\mathcal{K} \cong C(\mathbb{T})$ by an isomorphism that sends $u_n + \mathcal{K}$ to z. Let $\pi\colon T_n \longrightarrow C(\mathbb{T})$ denote the corresponding quotient map.

2.2. Choose a rank 1 projection $e \in \mathcal{K}$, so that $1 - u_n u_n^* \sim n \cdot e$. Since $u_n u_n^* \sim 1$, it follows that $1 \sim 1 \oplus n \cdot e$. Hence, $n[e] = 0$ in $K_0(T_n)$.

The groups $K_1\big(C(\mathbb{T})\big)$, $K_0(\mathcal{K})$, and $K_0\big(C(\mathbb{T})\big)$ are all infinite cyclic, generated by the classes $[z]$, $[e]$, and $[1_{C(\mathbb{T})}]$ respectively. Since $K_1(\mathcal{K}) = 0$, the standard 6-term exact sequence for the extension $0 \longrightarrow \mathcal{K} \xrightarrow{j} T_n \xrightarrow{\pi} C(\mathbb{T}) \longrightarrow 0$ looks as follows:

$$\begin{tikzcd}
K_0(\mathcal{K}) \arrow[r, "j_*"] & K_0(T_n) \arrow[r, "\pi_*"] & K_0\big(C(\mathbb{T})\big) \arrow[d] \\
K_1\big(C(\mathbb{T})\big) \arrow[u, "\partial"] & K_1(T_n) \arrow[l] & 0 \arrow[l]
\end{tikzcd}$$

In particular, π_* is surjective, and hence $K_0(T_n) = \mathbb{Z}[1_{T_n}] \oplus \ker(\pi_*)$. Further, $\ker(\pi_*) = \operatorname{im}(j_*)$ is generated by the class $[e] \in K_0(T_n)$. Since the unitary $z \in C(\mathbb{T})$ lifts to the isometry $u_n \in T_n$, we compute that $\partial([z]) = [1 - u_n^* u_n] - [1 - u_n u_n^*] = -n[e]$ (*cf.* [1, Section 8.3.2]). Hence, ∂ is injective and its image is generated by $n[e] \in K_0(\mathcal{K})$. It follows that $K_1(T_n) = 0$ and that

(1) $$[e] \in K_0(T_n) \text{ has order } n.$$

In particular, $K_0(T_n) = \mathbb{Z}[1_{T_n}] \oplus \mathbb{Z}[e] \cong \mathbb{Z} \oplus (\mathbb{Z}/n\mathbb{Z})$.

2.3. The next step is to manufacture a stably finite algebra with a quotient isomorphic to T_n. Moncasi used a trick of G. Bergman's (*cf.* [11, Example 5.10] and [18, Lemma 13]) for this purpose. The analog in our setting is the algebra

$$B_n = \{(x,y) \in T_n \oplus T_n^{\mathrm{op}} \mid \pi(x) = \pi(y)\},$$

where T_n^{op} denotes the opposite algebra of T_n (the C^*-algebra with the same elements, linear structure, adjoint, and norm as T_n, but with the reverse multiplication). Here π is also used to denote the quotient map $T_n^{\mathrm{op}} \longrightarrow C(\mathbb{T})^{\mathrm{op}} = C(\mathbb{T})$, and the commutativity of $C(\mathbb{T})$ is crucial to ensure that B_n is closed under the multiplication in $T_n \oplus T_n^{\mathrm{op}}$. The algebra B_1 was examined by Menal [17], who showed that it is finite and has stable rank 2.

We shall give a different description of B_n that is more natural in the C^*-algebra setting. Note that $\mathcal{B}(\ell^2)^{\mathrm{op}} \cong \mathcal{B}(\ell^2)$ via transposition of matrices (with respect to the standard orthonormal basis of ℓ^2). The transpose sends u_n to u_n^*, and so it restricts to an isomorphism of T_n^{op} onto T_n. Thus, we may replace T_n^{op} by T_n, provided we also replace $C(\mathbb{T})^{\mathrm{op}}$ by a copy of $C(\mathbb{T})$ in which z is replaced by z^*. We account for the latter using the automorphism θ of $C(\mathbb{T})$ given by $f(t) \longmapsto f(t^{-1})$; note that $\theta(z) = z^*$.

Thus, we now present B_n in the form

$$B_n = \{(x,y) \in T_n \oplus T_n \mid \pi(x) = \theta\pi(y)\},$$

which is a separable, unital sub-C^*-algebra of $T_n \oplus T_n$. (In this form, the case $n = 1$ yields an algebra that has been analyzed by Pedersen in [20, 9.3–9.6].) The subset $I = \mathcal{K} \oplus \mathcal{K}$ is a closed ideal of B_n, and $B_n/I \cong C(\mathbb{T})$. Further, the projections of $T_n \oplus T_n$ onto its factors induce isomorphisms $B_n/(0 \oplus \mathcal{K}) \cong T_n \cong B_n/(\mathcal{K} \oplus 0)$. There are two natural quotient maps $B_n \longrightarrow C(\mathbb{T})$, namely $(x,y) \longmapsto \pi(x)$ and $(x,y) \longmapsto \pi(y) = \theta\pi(x)$. We shall use π' to denote the former; thus $\pi'(u_n, u_n^*) = z$. Note that B_n is nuclear, since nuclearity is preserved in extensions [1, Theorem 15.8.2].

2.4. To compute the K-theory of B_n, we appeal to the standard 6-term exact sequence for the extension $0 \longrightarrow I \xrightarrow{j'} B_n \xrightarrow{\pi'} C(\mathbb{T}) \longrightarrow 0$, namely

$$\begin{tikzcd}
K_0(I) \arrow[r, "j'_*"] & K_0(B_n) \arrow[r, "\pi'_*"] & K_0\big(C(\mathbb{T})\big) \arrow[d] \\
K_1\big(C(\mathbb{T})\big) \arrow[u, "\partial'"] & K_1(B_n) \arrow[l] & 0 \arrow[l]
\end{tikzcd}$$

As before, $K_0(B_n) = \mathbb{Z}[1_{B_n}] \oplus \ker(\pi'_*)$, but this time $\ker(\pi'_*)$ is generated by the classes $[(e,0)]$ and $[(0,e)]$ in $K_0(B_n)$. In B_n, the element (u_n, u_n^*) is a partial isometry such that $(u_n, u_n^*)(u_n, u_n^*)^* = (u_n u_n^*, 1)$ and $(u_n, u_n^*)^*(u_n, u_n^*) = (1, u_n u_n^*)$; hence, $(u_n u_n^*, 1) \sim (1, u_n u_n^*)$. Since $(1 - u_n u_n^*, 0) \sim n \cdot (e,0)$ and $(0, 1 - u_n u_n^*) \sim n \cdot (0,e)$, we also have

$$n \cdot (e,0) \oplus (u_n u_n^*, 1) \sim (1,1) \sim n \cdot (0,e) \oplus (1, u_n u_n^*),$$

and thus $n[(e,0)] = n[(0,e)]$ in $K_0(B_n)$. Since K_0 of the left-hand projection $B_n \longrightarrow T_n$ sends $[(e,0)] - [(0,e)]$ to $[e]$, it follows from (1) that $k\big([(e,0)] - [(0,e)]\big) \neq 0$ for $k = 1, \dots, n-1$. Therefore

$$[(e,0)] - [(0,e)] \text{ has order } n \text{ in } K_0(B_n). \tag{2}$$

Since the unitary $z \in C(\mathbb{T})$ lifts to the partial isometry $(u_n, u_n^*) \in B_n$, we have

$$\begin{aligned} \partial'([z]) &= [1 - (u_n, u_n^*)^*(u_n, u_n^*)] - [1 - (u_n, u_n^*)(u_n, u_n^*)^*] \\ &= [(0, 1 - u_n u_n^*)] - [(1 - u_n u_n^*, 0)] = n[(0,e)] - n[(e,0)] \end{aligned} \tag{3}$$

in $K_0(I)$. Observe that $K_0(I) \cong \mathbb{Z}^2$ with basis $[(e,0)]$, $[(0,e)]$. In view of (3), it follows that ∂' is injective, and that its image is generated by the element $n\big([(e,0)] - [(0,e)]\big)$ in $K_0(I)$. Hence,

$$K_1(B_n) = 0, \tag{4}$$

and $\ker(j'_*) = \operatorname{im}(\partial')$ is contained in the subgroup $\mathbb{Z}\big([(e,0)] - [(0,e)]\big)$ of $K_0(I)$. This subgroup is a direct summand with complement $\mathbb{Z}[(0,e)]$. Consequently,

$$K_0(B_n) = \mathbb{Z}[1_{B_n}] \oplus \mathbb{Z}[(0,e)] \oplus \mathbb{Z}\big([(e,0)] - [(0,e)]\big) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus (\mathbb{Z}/n\mathbb{Z}). \tag{5}$$

(*Cf.* [20, Remark 9.6] for the case $n = 1$.) Under this isomorphism, the class $[1_{B_n}] \in K_0(B_n)$ corresponds to the triple $(1,0,0)$.

Observe that $K_0(I)^+ = \mathbb{Z}^+[(e,0)] + \mathbb{Z}^+[(0,e)]$, and so $\operatorname{im}(\partial') \cap K_0(I)^+ = \{0\}$. Since I and $C(\mathbb{T})$ are both stably finite, it now follows from [24, Lemma 1.5] that

$$B_n \text{ is stably finite.} \tag{6}$$

2.5. The order structure on $K_0(B_n)$ remains to be computed. We claim that in terms of the isomorphism (5),

$$\begin{aligned} K_0(B_n)^+ \cong \{(a,b,c+n\mathbb{Z}) \in \mathbb{Z} \oplus \mathbb{Z} \oplus (\mathbb{Z}/n\mathbb{Z}) \mid a > 0\} \\ \cup \{(0,b,c+n\mathbb{Z}) \in \mathbb{Z} \oplus \mathbb{Z} \oplus (\mathbb{Z}/n\mathbb{Z}) \mid b \geq c \geq 0\}. \end{aligned} \tag{7}$$

Consider an arbitrary element $p = a[1_{B_n}] + b[(0,e)] + c\big([(e,0)] - [(0,e)]\big)$ in $K_0(B_n)$, where a, b, c are integers. Since π'_* sends p to $a[1_{C(\mathbb{T})}]$, we see that if $p \in K_0(B_n)^+$ then $a \geq 0$. On the other hand, $|c| \cdot (e,0) \oplus |b-c| \cdot (0,e)$ is equivalent to a projection in I, whence

$$-b[(0,e)] - c\big([(e,0)] - [(0,e)]\big) \leq |c| \cdot [(e,0)] + |b-c| \cdot [(0,e)] \leq [1_{B_n}]$$

in $K_0(B_n)$. Thus $p \in K_0(B_n)^+$ in case $a > 0$.

Now assume that $a = 0$. If $p \in K_0(B_n)^+$, then $p = [g]$ for some projection $g \in M_\infty(B_n)$, and $b \cdot (0,e) \oplus c \cdot (e,0) \oplus m \cdot 1_{B_n} \sim c \cdot (0,e) \oplus g \oplus m \cdot 1_{B_n}$ for some positive integer m. Since B_n/I is stably finite, g must lie in $M_\infty(I)$. Thus, the elements of the form $p = b[(0,e)] + c\big([(e,0)] - [(0,e)]\big)$ in $K_0(B_n)^+$ are precisely the $K_0(B_n)$-classes of projections from $M_\infty(I)$. Any such projection is equivalent to $c \cdot (e,0) \oplus d \cdot (0,e)$ for some nonnegative integers c, d, and

$$c[(e,0)] + d[(0,e)] = (c+d)[(0,e)] + c\big([(e,0)] - [(0,e)]\big).$$

Hence, $j'_*(K_0(I)^+) = \big\{b[(0,e)] + c\big([(e,0)] - [(0,e)]\big) \mid b \geq c \geq 0\big\}$, and the claim is verified. Thus $K_0(B_n)$ is isomorphic to the lexicographic direct sum of $\mathbb{Z}$ with the ordered group

$$G_n := \mathbb{Z} \oplus (\mathbb{Z}/n\mathbb{Z}), \quad \text{where} \quad G_n^+ := \{(b, c+n\mathbb{Z}) \mid b \geq c \geq 0\}.$$

2.6. The algebra B_n is the C^*-algebra analog of Moncasi's example in [19], and has the same ordered K_0, which (as is easily checked) is not a Riesz group. In Moncasi's construction, the role of $C(\mathbb{T})$ is taken by a field (rational functions in one variable), which ensures that his example is a von Neumann regular ring. In the present setting, however, we do not yet have a large enough supply of projections, since $C(\mathbb{T})$ does not have real rank zero. To manufacture an example with real rank zero, we form an inductive limit similar to those studied in [13], using B_n in place of $C(X)$ as our basic building block.

Choose a dense sequence of points $\{t_1, t_2, \ldots\} \subset \mathbb{T}$. For $k = 1, 2, \ldots$, let

$$\delta_k \colon B_n \longrightarrow C(\mathbb{T}) \longrightarrow \mathbb{C} \longrightarrow B_n$$

denote the composition of the quotient map $\pi' \colon B_n \to C(\mathbb{T})$ with evaluation at t_k, followed by the unital embedding $\mathbb{C} \hookrightarrow B_n$. We shall also use δ_k to denote the induced maps $M_\bullet(B_n) \to M_\bullet(B_n)$. Set A_n equal to the C^*-inductive limit of the sequence

$$B_n \xrightarrow{\phi_1} M_2(B_n) \xrightarrow{\phi_2} M_4(B_n) \xrightarrow{\phi_3} \cdots$$

where the connecting maps $\phi_k: M_{2^{k-1}}(B_n) \longrightarrow M_{2^k}(B_n)$ are block diagonal maps given by

$$\phi_k(x) = \begin{pmatrix} x & 0 \\ 0 & \delta_k(x) \end{pmatrix}. \tag{8}$$

Obviously A_n is separable, unital, and nuclear [1, Theorem 15.8.2]. Further, because of (6) we see that

$$A_n \text{ is stably finite.} \tag{9}$$

Since all the δ_k vanish on I, each ϕ_k maps $M_{2^{k-1}}(I)$ into $M_{2^k}(I)$. The inductive limit of these ideals yields a closed ideal J in A_n, with $J \cong I \otimes \mathcal{K} \cong I$. Thus J is AF; in particular, J has real rank zero and $K_1(J) = 0$. On the other hand, A_n/J is isomorphic to the inductive limit of a sequence

$$C(\mathbb{T}) \longrightarrow M_2\big(C(\mathbb{T})\big) \longrightarrow M_4\big(C(\mathbb{T})\big) \longrightarrow \cdots$$

with connecting maps analogous to (8). By [13, Lemma 1, Theorems 3,9], A_n/J is simple with stable rank 1 and real rank 0. Therefore

$$A_n \text{ has real rank } 0 \tag{10}$$

by [27, Proposition 2.3; 28, Corollary 2.12] (*cf.* [3, Theorem 3.14, Corollary 3.16]). Note also that

$$K_1(A_n) = 0, \tag{11}$$

in view of (4) and the continuity of K_1.

2.7. The inductive limits of the ideals $M_{2^{k-1}}(\mathcal{K} \oplus 0)$ and $M_{2^{k-1}}(0 \oplus \mathcal{K})$ yield closed ideals J_1 and J_2 in A_n such that $J_1 + J_2 = J$. Further, T_n, which is isomorphic to $B_n/(\mathcal{K} \oplus 0)$ and to $B_n/(0 \oplus \mathcal{K})$, embeds in A_n/J_1 and in A_n/J_2. Hence, A_n/J_1 and A_n/J_2 are infinite. It follows, in particular, that A_n cannot have stable rank 1. On the other hand, since J and A_n/J have stable rank 1, the stable rank of A_n is at most 2 [25, Theorem 4; 22, Corollary 4.12]. Therefore

$$A_n \text{ has stable rank } 2. \tag{12}$$

To compute $K_0(A_n)$, identify each $K_0\big(M_{2^{k-1}}(B_n)\big)$ with $K_0(B_n)$, equipped with the order-unit $2^{k-1}[1_{B_n}]$. Then each connecting map $K_0(\phi_k)$ doubles $[1_{B_n}]$ while fixing $[(e, 0)]$ and $[(0, e)]$. In view of (5) and (7), it follows that

$$K_0(A_n) \cong \mathbb{Z}\left[\frac{1}{2}\right] \bigoplus_{\text{lex}} G_n \tag{13}$$

as ordered groups; further, the class $[1_{A_n}] \in K_0(A_n)$ corresponds to the pair $(1, 0)$ under this isomorphism. We conclude by observing the same failure of Riesz decomposition noted by Moncasi, in case $n \geq 2$. Namely, the positive elements $x = (0, 1, 0)$, $y_1 = (0, 1, 1 + n\mathbb{Z})$, and $y_2 = (0, n-1, n-1 + n\mathbb{Z})$ satisfy $x \leq y_1 + y_2$ but $x \not\leq y_1$ and $x \not\leq y_2$.

Since x is an atom in G_n^+, it cannot be the sum of two positive elements dominated by y_1 and y_2. Therefore

(14) $K_0(A_n)$ is not a Riesz group when $n \geq 2$.

2.8. In the case $n = 1$, the statement (14) turns from negative to positive, since

$$K_0(A_1) \cong \mathbb{Z}\left[\frac{1}{2}\right] \bigoplus_{\text{lex}} \mathbb{Z},$$

which is a dimension group [12, Proposition 3.3]. Thus A_1 has the same K-theory as an AF-algebra, *i.e.*, there exists an AF-algebra A' with $K_0(A') \cong \mathbb{Z}[\frac{1}{2}] \oplus_{\text{lex}} \mathbb{Z}$ as ordered groups (see [8, Theorem 5.5] and [7, Theorem 2.2], or [1, Theorem 7.4.1]). However, A_1 is not AF, by virtue of (12).

Another difference between A_1 and the AF-algebra A' is that A' has only one proper nonzero closed ideal (corresponding to the single proper nonzero ideal $0 \oplus \mathbb{Z}$ in $\mathbb{Z}[\frac{1}{2}] \oplus_{\text{lex}} \mathbb{Z}$), whereas A_1 has three (namely J_1, J_2, J). The ideals J_1 and J_2 are not seen by $K_0(A_1)$ since the ideals of $K_0(A_1)$ correspond just to the stably cofinite ideals of A_1 [14, Theorem 10.9].

2.9. The algebras A_n are also interesting from the point of view of extensions. First note that A_n/J is just the basic Bunce-Deddens algebra of type 2^∞. (For instance, Elliott's invariant $K_*(A_n/J)$ is clearly the graded ordered group $\mathbb{Z}[\frac{1}{2}] \oplus \mathbb{Z}$ with positive cone $\{(0,0)\} \cup \{(a,b) \mid a > 0\}$ and order-unit $(1,0)$, which is the same as for the Bunce-Deddens algebra, and [10, Theorem 7.1] applies.) Thus

(15) A_n is an extension of the Bunce-Deddens algebra by $\mathcal{K} \oplus \mathcal{K}$.

Since A_n/J_1 is infinite, it cannot be approximately homogeneous (*i.e.*, it is not an inductive limit of finite direct products of matrix algebras over commutative C^*-algebras). It follows immediately that

(16) A_n is not approximately homogeneous.

That extensions of Bunce-Deddens algebras by AF algebras need not be approximately homogeneous was already known via an example of Putnam analyzed in [16, Section 1, end] and [2, Section 1, end]. These examples show that the class of inductive limit algebras analyzed in [10] is not closed under extensions. There also exist extensions of algebras in this class which have stable rank 1 yet are not approximately homogeneous [6, Example 4.5].

3. **Summary.**

THEOREM 3.1. *(a) The algebras A_n are stably finite, unital, separable, nuclear C^*-algebras with real rank* 0 *and stable rank* 2.

(b) $K_1(A_n) = 0$ *and* $K_0(A_n) \cong \mathbb{Z}[\frac{1}{2}] \oplus_{\text{lex}} G_n$*, where* G_n *denotes the partially ordered group* $\mathbb{Z} \oplus (\mathbb{Z}/n\mathbb{Z})$ *with positive cone* $G_n^+ = \{(b, c + n\mathbb{Z}) \mid b \geq c \geq 0\}$*. Under this isomorphism, the class* $[1_{A_n}] \in K_0(A_n)$ *corresponds to the element* $(1,0) \in \mathbb{Z}[\frac{1}{2}] \oplus_{\text{lex}} G_n$*.*

(c) If $n \geq 2$*, then* $K_0(A_n)$ *fails to satisfy the Riesz decomposition property.*

(d) There exists a unital approximately finite dimensional C^**-algebra* A' *such that* $\big(K_0(A'), [1_{A'}]\big) \cong \big(K_0(A_1), [1_{A_1}]\big)$ *(and* $K_1(A') = 0 = K_1(A_1)$*). However,* $A' \not\cong A_1$*, because* A_1 *is not approximately finite dimensional.*

(e) A_n *is an extension of the Bunce-Deddens algebra by* $\mathcal{K} \oplus \mathcal{K}$*, but* A_n *is not approximately homogeneous.* ■

3.2. Dădarlăt has pointed out that the existence of C^*-algebras with most of the above properties can be predicted from the Universal Coefficient Theorem of Rosenberg and Schochet [23, Theorem 1.17; 1, Theorem 23.1.1]. Namely, let $\mathcal{B}$ be the Bunce-Deddens algebra, and let f_n be the group homomorphism from $K_1(\mathcal{B}) \cong \mathbb{Z}$ to $K_0(\mathcal{K} \oplus \mathcal{K}) \cong \mathbb{Z}^2$ with matrix $(-n, n)$. By the UCT, there is an element of $KK(\mathcal{B}, \mathcal{K} \oplus \mathcal{K})$ inducing f_n, and this KK-element may be represented by a unital extension $0 \longrightarrow \mathcal{K} \oplus \mathcal{K} \longrightarrow \mathcal{A}_n \longrightarrow \mathcal{B} \longrightarrow 0$ for which the index map $K_1(\mathcal{B}) \longrightarrow K_0(\mathcal{K} \oplus \mathcal{K})$ equals f_n. As above, it follows from [27, 28] that $\mathcal{A}_n$ has real rank zero, from [24] that $\mathcal{A}_n$ is stably finite, and from the 6-term exact sequence that $K_1(\mathcal{A}_n) = 0$ and $K_0(\mathcal{A}_n) \cong \mathbb{Z}[\frac{1}{2}] \oplus \mathbb{Z} \oplus (\mathbb{Z}/n\mathbb{Z})$.

It does not appear that one can completely determine the order structure on $K_0(\mathcal{A}_n)$. However, enough can be determined to see the failure of Riesz decomposition when $n \geq 2$, as follows. Let H_n denote the kernel of K_0 of the quotient map $\mathcal{A}_n \longrightarrow \mathcal{B}$. Since $\mathcal{B}$ is stably finite, H_n is an ideal of $K_0(\mathcal{A}_n)$ [14, Lemma 10.5], and so it suffices to show that Riesz decomposition fails in H_n. It is easily checked that $H_n \cap K_0(\mathcal{A}_n)^+$ equals the image of $K_0(\mathcal{K} \oplus \mathcal{K})^+$ (using again the fact that $\mathcal{B}$ is stably finite). Thus H_n is isomorphic to the quotient of $\mathbb{Z}^2$ (with the product ordering) modulo the image of f_n. In other words, $H_n \cong G_n$ as ordered groups, and we conclude as in (2.7) that H_n does not satisfy Riesz decomposition unless $n = 1$.

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Canad. Math. Bull. Vol. 39 (4), 1996 pp. 438–447

A GENERALIZATION OF AN INEQUALITY OF BHATTACHARYA AND LEONETTI

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ABSTRACT. We show that bounded John domains and bounded starshaped domains with respect to a point satisfy the following inequality

$$\int_D F\Big(b\frac{|u(x)-u_D|}{\operatorname{dia}(D)}\Big)\,dx \le K\int_D F\big(|\nabla u(x)|\big)\,dx,$$

where $F\colon[0,\infty)\longrightarrow[0,\infty)$ is a continuous, convex function with $F(0)=0$, and u is a function from an appropriate Sobolev class. Constants b and K do depend at most on D. If $F(x)=x^p$, $1\le p<\infty$, this inequality reduces to the ordinary Poincaré inequality.

1. **Introduction.** Tilak Bhattacharya and Francesco Leonetti introduced the following version of the Poincaré inequality

$$\int_D F\Big(b\frac{|u(x)-u_D|}{\operatorname{dia}(D)}\Big)\,dx \le K\int_D F(|\nabla u(x)|)\,dx. \tag{1.1}$$

Here, $F\colon[0,\infty)\longrightarrow[0,\infty)$ is a convex, continuous function with $F(0)=0$, D is a bounded domain in R^n, u is a function from an appropriate Sobolev class, and u_D stands for the integral average of u over D. Constants $b\in(0,1]$ and $K>0$ depend at most on D. A domain D is an *F-Poincaré domain*, write $D\in\mathcal{P}(F)$, whenever there are constants $b=b(D)$ and $K=K(D)$ such that (1.1) holds for all $u\in W_1^1(D)$ and $F(|\nabla u|)\in L^1(D)$.

Bhattacharya and Leonetti proved that inequality (1.1) with $b=1$ holds for convex domains, [1, Lemma 1], and with additional assumptions of F for an annulus, [1, Theorem 2]. In this paper we show that John domains and starshaped domains are F-Poincaré domains. Further we consider a modification of (1.1) in Section 6.

If $F(x)=x^p$, $1\le p<\infty$, inequality (1.1) reduces to the ordinary Poincaré inequality

$$\int_D |u(x)-u_D|^p\,dx \le \kappa_p(D)^p\int_D|\nabla u(x)|^p\,dx,$$

whenever $u\in W_p^1(D)$. It is customary to write $D\in\mathcal{P}(p)$ and $\kappa_p(D)=K^{1/p}\operatorname{dia}(D)\,b^{-1}$ and to say D is a p-Poincaré domain. Starshaped domains as well as John domains are p-Poincaré domains for all p, $1\le p<\infty$, [4, Theorem 3.1], [3, Theorems 3.1 and 8.5].

We restate the result of Bhattacharya and Leonetti here.

Received by the editors June 8, 1995.

AMS subject classification: Primary: 46E35; Secondary: 26D10.

Key words and phrases: Poincaré-type inequalities, John domains, starshaped domains.

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THEOREM 1.2 [1, Lemma 1]. *Let D be a convex, bounded subset of R^n, $n \geq 1$. Let $F\colon [0,\infty) \longrightarrow [0,\infty)$ be a continuous, convex function with $F(0) = 0$. If $u \in W_1^1(D)$ such that $F(|\nabla u|) \in L^1(D)$, then (1.1) holds with $b = 1$ and $K = (\frac{\omega_n \operatorname{dia}(D)^n}{|D|})^{1-1/n}$.*

A generalization of Theorem 1.2 is that John domains and starshaped domains are F-Poincaré domains, Theorems 4.1 and 5.1.

2. **Notation and definitions.** Throughout this paper we let D and G be bounded domains of euclidean n-space R^n, $n \geq 2$.

If $x \in R^n$ and $r > 0$, then $B^n(x,r) = \{y \in R^n \mid |x-y| < r\}$ is an open ball in R^n and the sphere $S^{n-1}(x,r)$ is its boundary. We use the abbreviations $B^n(r) = B^n(0,r)$ and $S^{n-1}(r) = S^{n-1}(0,r)$.

The euclidean distance between sets A and B is written as $d(A,B)$, and $d(x,\partial A)$ denotes the distance from $x \in A$ to the boundary of A. We let $\operatorname{dia}(A)$ denote the diameter of A. We write tQ for the cube with the same center as Q and dilated by a factor $t > 1$.

The average of a function u is $u_A = \frac{1}{|A|}\int_A u(x)\,dx = ⨍_A u(x)\,dx$ if $|A| > 0$; here $|A|$ stands for the n-dimensional Lebesgue measure of A. We write $|B^n(1)| = \omega_n$.

The L^p-norm of u in A is $\|u\|_{L^p(A)} = (\int_A |u(y)|^p\,dy)^{1/p}$. The Sobolev space $W_p^1(G)$, $1 \leq p < \infty$, is the space of functions $u \in L^p(G)$ whose first distributional partial derivatives belong to $L^p(G)$. In $W_p^1(G)$ we use the norm $\|u\|_{W_p^1(G)} = \|u\|_{L^p(G)} + \|\nabla u\|_{L^p(G)}$; here $\nabla u = (\partial_1 u, \ldots, \partial_n u)$ is the gradient of u.

We let $c(*,\ldots,*)$ denote a constant which depends only on the quantities appearing in the parentheses.

A domain D is called an (α,β)-*John domain*, $0 < \alpha \leq \beta < \infty$, if there is $x_0 \in D$ such that each $x \in D$ can be joined to x_0 by a rectifiable curve $\gamma\colon [0,\ell] \longrightarrow D$ parametrized by arc length with $\ell \leq \beta$ and

$$d\big(\gamma(t),\partial D\big) \geq \frac{\alpha}{\ell}t, \quad t \in [0,\ell].$$

Convex domains, Lipschitz domains, and bounded uniform domains are John domains. John domains form a proper subclass of domains satisfying a quasihyperbolic boundary condition. We refer to [2] for detailed discussion of these concepts.

A bounded domain in R^n is called *starshaped with respect to a point* $x_0 \in D$, if each ray starting from x_0 intersects ∂D exactly at one point. A starshaped domain is not necessarily a John domain: a simple example is

$$D = \{(x_1,x_2) \in B^2\big((1,0),1\big) : |x_2| < x_1^2\}.$$

The following chains and decompositions of a domain are essential to our sufficient condition in Theorem 3.1.

2.1. CHAINS. Sets D_i, $i = 0,1,\ldots,k$, in R^n form a *chain*, abbreviated $C(D_k) = (D_0,D_1,\ldots,D_k)$, if $D_i \cap D_j \neq \emptyset$ if and only if $|i-j| \leq 1$.

2.2. DECOMPOSITIONS. Let $\mathcal{W}$ be a family of domains $D \in \mathcal{P}(F)$ with $0 < b_0 \leq b(D)$ and $K(D) \leq c_0 < \infty$ such that $D \in \mathcal{P}(1)$ with $\kappa_1(D) \leq c_1 < \infty$. We call $\mathcal{W}$ an *F-Poincaré decomposition* of G, if there are constants c_2, c_3, and N with the following properties:

(i) $G = \bigcup_{D \in \mathcal{W}} D$,

(ii) $\sum_{D \in \mathcal{W}} \chi_D(x) \leq N\chi_G(x)$ for all $x \in R^n$, and

(iii) there is a domain $D_0 \in \mathcal{W}$ such that for each $D \in \mathcal{W}$ there is a chain $C(D) = (D_0, D_1, \dots, D_k)$ of domains in $\mathcal{W}$ with

$$\max\{|D_i|, |D_{i+1}|\} \leq c_2 |D_i \cap D_{i+1}| \tag{2.3}$$

for $i = 0, 1, \dots, k-1$; $D = D_k$; and

$$\sum_{A \in C(D)} \kappa_1(A) \leq c_3 b_0^{-1} c_2^{-1} \operatorname{dia}(G). \tag{2.4}$$

For each $D \in \mathcal{W}$ we fix a chain $C(D)$ satisfying (2.3) and (2.4) and call this chain the F-Poincaré chain from D_0 to D. For a fixed set $A \in \mathcal{W}$ we write

$$A(\mathcal{W}) = \{D \in \mathcal{W} \mid A \in C(D)\}.$$

If D in R^n is an (α, β)-John domain and W is a Whitney decomposition of D into Whitney cubes Q, [6, VI], then $\{\text{int } \frac{9}{8}Q \mid Q \in W\}$ forms an F-Poincaré decomposition of D, see the proof for Theorem 4.1.

3. A sufficient condition for a domain to be an F-Poincaré domain.

Our main result is the following theorem which gives a sufficient condition for a domain to be an F-Poincaré domain.

THEOREM 3.1. *Let $G \subset R^n$ be a bounded domain and let $\mathcal{W}$ be an F-Poincaré decomposition of G. Suppose that there are constants $b_1 < \infty$ and $\varepsilon \in [0, 1]$ such that*

$$\sum_{D \in A(\mathcal{W})} |D| \leq b_1 \, \kappa_1(A)^{-\varepsilon} |A| \tag{3.2}$$

for all $A \in \mathcal{W}$. Then $G \in \mathcal{P}(F)$.

PROOF FOR THEOREM 3.1. Since F is an increasing, convex, continuous function,

$$F\big(|u(x) - u_G|\big) \leq \frac{1}{2}\Big(F\big(2|u(x) - c|\big) + F(2|c - u_G|)\Big),$$

where by Jensen's inequality

$$F(2|u_G - c|) \leq F\Big(2 \fint_G |u(y) - c| \, dy\Big) \leq \fint_G F\big(2|u(y) - c|\big) \, dy;$$

here, $c \in R$. Hence

$$\int_G F\big(|u(x) - u_G|\big) \, dx \leq \int_G F\big(2\,|u(x) - c|\big) \, dx \tag{3.3}$$

for each $c \in R$. Thus we need to estimate $F(2|u(y) - c|)$ for some constant $c \in R$.

We apply a similar argument as in [3, Theorem 4.4]. Since $\mathcal{W}$ is an F-Poincaré decomposition of G, there is a domain $D_0 \in \mathcal{W}$ such that for each $D \in \mathcal{W}$ we can fix a chain satisfying (2.3) and (2.4). We will estimate

$$\begin{aligned} F\Big(\frac{b_0}{4c_3\,\mathrm{dia}(G)}|u(x) - u_{D_0}|\Big) \le \frac{1}{2}\Big(&F\Big(\frac{b_0}{2c_3\,\mathrm{dia}(G)}|u(x) - u_D|\Big) \\ &+ F\Big(\frac{b_0}{2c_3\,\mathrm{dia}(G)}|u_D - u_{D_0}|\Big)\Big). \end{aligned} \tag{3.4}$$

Recall $D \in \mathcal{P}(F)$ with $K(D) \le c_0 < \infty$ and especially $D \in \mathcal{P}(1)$ with $\kappa_1(D) \le c_1 < \infty$. Inequality (2.3) and the fact $D_j \in \mathcal{P}(1)$ yield

$$\begin{aligned} |u_{D_k} - u_{D_0}| &\le \sum_{j=0}^{k-1} |u_{D_j} - u_{D_{j+1}}| \\ &= \sum_{j=0}^{k-1} \fint_{D_j \cap D_{j+1}} |u_{D_j} - u_{D_{j+1}}|\,dx \\ &\le 2c_2 \sum_{j=0}^{k} \fint_{D_j} |u_{D_j} - u(x)|\,dx \\ &\le 2c_2 \sum_{j=0}^{k} \kappa_1(D_j) \fint_{D_j} |\nabla u(x)|\,dx. \end{aligned}$$

Thus using (2.4), convexity, and Jensen's inequality we obtain

$$\begin{aligned} F\Big(\frac{b_0 c_2}{2c_3\,\mathrm{dia}(G)}|u_D - u_{D_0}|\Big) &\le F\Big(\frac{b_0 c_2}{c_3\,\mathrm{dia}(G)} \sum_{A \in C(D)} \kappa_1(A) \fint_A |\nabla u(y)|\,dy\Big) \\ &\le F\Big(\sum_{A \in C(D)} \frac{\kappa_1(A)}{\sum_{B \in C(D)} \kappa_1(B)} \fint_A |\nabla u(y)|\,dy\Big) \\ &\le \sum_{A \in C(D)} \frac{\kappa_1(A)}{\sum_{B \in C(D)} \kappa_1(B)} F\Big(\fint_A |\nabla u(y)|\,dy\Big) \\ &\le \sum_{A \in C(D)} \frac{\kappa_1(A)}{\kappa_1(D_0)} \fint_A F\big(|\nabla u(y)|\big)\,dy. \end{aligned} \tag{3.5}$$

Inequalities (3.3)–(3.5) imply

$$\begin{aligned} &\int_G F\Big(\frac{b_0}{8c_3\,\mathrm{dia}(G)}|u(x) - u_G|\Big)\,dx \\ &\qquad \le \frac{1}{2} \sum_{D \in \mathcal{W}} \int_D F\Big(\frac{b(D)}{2c_3\,\mathrm{dia}(D)}|u(x) - u_D|\Big)\,dx \\ &\qquad\quad + \frac{1}{2} \sum_{D \in \mathcal{W}} |D| \sum_{A \in C(D)} \frac{\kappa_1(A)}{\kappa_1(D_0)} \fint_A F\big(|\nabla u(y)|\big)\,dy. \end{aligned} \tag{3.6}$$

We may assume that $1 \le c_3$. Since $D \in \mathcal{P}(F)$, inequality (1.1) yields

$$\begin{aligned} \sum_{D\in\mathcal{W}} \int_D F\Big(\frac{b(D)}{c_3\,\mathrm{dia}(D)}|u(x)-u_D|\Big)\,dx &\le c_0 \sum_{D\in\mathcal{W}} \int_D F\big(|\nabla u(x)|\big)\,dx \\ &\le Nc_0 \int_G F\big(|\nabla u(x)|\big)\,dx. \end{aligned} \tag{3.7}$$

Rearranging the double sum in (3.6), and using (3.2) and the inequality $\kappa_1(A) \le c_1$ we obtain

$$\begin{aligned} &\sum_{D\in\mathcal{W}} \sum_{A\in C(D)} |D|\kappa_1(A) \fint_A F\big(|\nabla u(y)|\big)\,dy \\ &\qquad = \sum_{A\in\mathcal{W}} \sum_{D\in A(\mathcal{W})} |D|\kappa_1(A) \fint_A F\big(|\nabla u(y)|\big)\,dy \\ &\qquad \le b_1 \sum_{A\in\mathcal{W}} \kappa_1(A)^{1-\varepsilon} \int_A F\big(|\nabla u(y)|\big)\,dy \\ &\qquad \le b_1 c_1^{1-\varepsilon} N \int_G F\big(|\nabla u(x)|\big)\,dx. \end{aligned} \tag{3.8}$$

Substituting (3.7) and (3.8) into (3.6) implies the desired inequality

$$\int_G F\Big(\frac{b_0}{8c_3\,\mathrm{dia}(G)}|u(x)-u_G|\Big)\,dx \le c(b_1,c_0,c_1,\varepsilon,N) \int_G F\big(|\nabla u(x)|\big)\,dx$$

and Theorem 3.1 is proved.

4. **John domains.** Applying Theorem 3.1 and its proof to a John domain yields

THEOREM 4.1. *An (α,β)-John domain D in R^n is an F-Poincaré domain with $b = c(n)\frac{\alpha}{\beta}$ and $K = c(n)(\frac{\beta}{\alpha})^{n+1}$.*

PROOF. Let $\mathcal{W}$ be a Whitney decomposition of D into cubes Q. Theorem 1.2 yields that $K(Q) = c(n) := c_0$ and $\kappa_1(Q) = c(n)\,\mathrm{dia}(Q) := c_1$. Fix $Q_0 \in \mathcal{W}$ with $x_0 \in Q_0$. The John property in a domain means that for each $Q \in \mathcal{W}$ there is a chain $C(\mathrm{int}\,\frac{9}{8}Q)$ of cubes $\mathrm{int}\,\frac{9}{8}Q_j$ $Q_j \in \mathcal{W}, j = 0,1,\dots,k$, $Q = Q_k$, such that

$$\sum_{j=i}^{k} \mathrm{dia}(Q_j) \le c(n)\frac{\beta}{\alpha}\,\mathrm{dia}(Q_i) \le c(n)\frac{\beta}{\alpha}\,\mathrm{dia}(D)$$

for all $i = 0,1,\dots,k$, see [3, proofs for Proposition 6.1 and Lemma 8.3]. Hence there are constants $c_2 = c_2(n)$ and $c_3 = c_3(n)\frac{\beta}{\alpha}$ such that (2.3) and (2.4) are true.

The above result combined to the fact that there are not too many Whitney cubes of the same size in a John domain yields that for all $A \in \mathcal{W}' = \{\mathrm{int}\,\frac{9}{8}Q \mid Q \in \mathcal{W}\}$

$$\begin{aligned} \sum_{\mathrm{int}\frac{9}{8}Q\in A(\mathcal{W}')} |Q| &\le \sum_{j=1}^{\infty} \sum_{Q_j\in S} |Q_j| \\ &\le \sum_{j=1}^{\infty} c(n)\Big(\frac{\beta}{\alpha}\Big)^n \Big(\frac{\alpha}{c(n)\beta}\Big)^{\delta} 2^{-j\delta}|A| \\ &\le c(n)\Big(\frac{\beta}{\alpha}\Big)^n |A|, \end{aligned}$$

where $S = \left\{\operatorname{int}\frac{9}{8}Q : \frac{\alpha\,\mathrm{dia}(A)}{2^j\beta c(n)} \leq \mathrm{dia}(\frac{9}{8}Q) \leq 2\frac{\alpha\,\mathrm{dia}(A)}{2^j\beta c(n)}\right\}$ and $\delta = \delta(n, \frac{\alpha}{\beta})$, see [3, Lemma 8.4]. Now $\{\operatorname{int}\frac{9}{8}Q \mid Q \in \mathcal{W}\}$ is an F-Poincaré decomposition of D and (3.2) is satisfied, when $\varepsilon = 0$. Thus $D \in \mathcal{P}(F)$ by Theorem 3.1.

The proof for the Theorem 3.1 gives that $b = c(n)\frac{\alpha}{\beta}$ and

$$K = c(n)\left(1 + \left(\frac{\beta}{\alpha}\right)^n \frac{\mathrm{dia}(D)}{\mathrm{dia}(Q_0)}\right) \leq \left(\frac{\beta}{\alpha}\right)^{n+1},$$

since $\alpha \leq \mathrm{dia}(Q_0) \leq \mathrm{dia}(D) \leq \beta$.

5. **Starshaped domains.**

THEOREM 5.1. *If D in R^n is a domain which is starshaped with respect to a point x_0, then $D \in \mathcal{P}(F)$. Here, $b = \frac{1}{18}$ and $K = K\big(n, d(x_0, \partial D), \max_{x\in\partial D} d(x, x_0)\big)$.*

We need the following trace lemma for the proof of Theorem 5.1.

LEMMA 5.2. *Let D be a domain in R^n and let $B^n(2\ell) \subset D$. If $F\colon [0,\infty) \to [0,\infty)$ is a convex, continuous function with $F(0) = 0$, then*

$$\int\limits_{S^{n-1}(\xi)} F\Big(\frac{1}{2\ell}|u(z)|\Big)\, dm_{n-1}(z) \leq \frac{1}{\ell}\int\limits_{B^n(\ell)} F\Big(\frac{1}{\ell}|u(x)|\Big)\, dx + \frac{2^{n-1}}{\ell}\int\limits_{B^n(\ell)} F\big(|\nabla u(x)|\big)\, dx,$$

for each $\xi \in [\ell/2, \ell]$ whenever $u \in C^1(D)$.

PROOF. By the mean value theorem for integrals we have

$$\begin{aligned}(5.3)\qquad &\int\limits_{r=\ell/2}^{\ell}\int\limits_{S^{n-1}(1)} F\big(|u(\theta, r)|\big)\, r^{n-1}\, dm_{n-1}(\theta)\, dr\\ &\qquad = (\ell - \ell/2)\int\limits_{S^{n-1}(1)} F\big(|u(\theta,\sigma)|\big)\,\sigma^{n-1}\, dm_{n-1}(\theta)\end{aligned}$$

for some $\sigma \in [\ell/2, \ell]$.

On the other hand for ξ with $\ell/2 \leq \xi \leq \sigma \leq \ell$,

$$\begin{aligned}F\Big(\frac{1}{2\ell}|u(\theta,\xi)|\Big) &\leq F\Big(\frac{1}{2\ell}|u(\theta,\sigma)| + \frac{1}{2\ell}\int_\xi^\sigma |D_r u(\theta,t)|\, dt\Big)\\ &\leq \frac{1}{2}F\Big(\frac{1}{\ell}|u(\theta,\sigma)|\Big) + \frac{1}{2}\frac{1}{\ell/2}\int_{\ell/2}^{\ell} F\big(|D_r u(\theta,t)|\big)\, dt.\end{aligned}$$

Hence

$$\begin{aligned}&F\Big(\frac{1}{2\ell}|u(\theta,\xi)|\Big)\xi^{n-1}\\ &\qquad\leq \frac{1}{2}F\Big(\frac{1}{\ell}|u(\theta,\sigma)|\Big)\sigma^{n-1} + \frac{1}{\ell}\Big(\frac{2}{\ell}\Big)^{n-1}\ell^{n-1}\int_{\ell/2}^{\ell} F\big(|D_r u(\theta,t)|\big)t^{n-1}dt,\end{aligned}$$

where we have used $\xi^{n-1} \leq \sigma^{n-1}$ and $1 \leq t^{n-1}/(\ell/2)^{n-1}$.

Combining the estimates and (5.3) we obtain

$$
\begin{aligned}
&\int\limits_{S^{n-1}(\xi)} F\Big(\frac{1}{2\ell}|u(z)|\Big)\,dm_{n-1}(z)\\
&\qquad = \int\limits_{S^{n-1}(1)} F\Big(\frac{1}{2\ell}|u(\theta,\xi)|\Big)\xi^{n-1}\,dm_{n-1}(\theta)\\
&\qquad \le \frac{1}{2}\int\limits_{S^{n-1}(1)} F\Big(\frac{1}{\ell}|u(\theta,\sigma)|\Big)\sigma^{n-1}\,dm_{n-1}(\theta)\\
&\qquad\qquad + \frac{2^{n-1}}{\ell}\int\limits_{t=\ell/2}^{\ell}\int\limits_{S^{n-1}(1)} F\big(|D_r u(\theta,t)|\big)t^{n-1}\,dm_{n-1}(\theta)\,dt\\
&\qquad \le \frac{1}{\ell}\int\limits_{r=\ell/2}^{\ell}\int\limits_{S^{n-1}(1)} F\Big(\frac{1}{\ell}|u(\theta,r)|\Big)r^{n-1}\,dm_{n-1}(\theta)\,dr\\
&\qquad\qquad + \frac{2^{n-1}}{\ell}\int\limits_{B^n(\ell)\setminus B^n(\ell/2)} F\big(|\nabla u(x)|\big)\,dx\\
&\qquad = \frac{1}{\ell}\int\limits_{B^n(\ell)\setminus B^n(\ell/2)} F\Big(\frac{1}{\ell}|u(x)|\Big)\,dx\\
&\qquad\qquad + \frac{2^{n-1}}{\ell}\int\limits_{B^n(\ell)\setminus B^n(\ell/2)} F\big(|\nabla u(x)|\big)\,dx.
\end{aligned}
$$

This yields the desired inequality and the proof is complete.

PROOF FOR THEOREM 5.1. Write $d(x_0,\partial D) = 2\ell$, $\max_{x\in\partial D} d(x,x_0) = L$, and $B^n(x_0,\frac{1}{2}\ell) = B$. It suffices to consider functions $u \in W_p^1(D)\cap C^\infty(D)$, *cf.* [5, Theorem 1.1.6/1]. We assume, for convenience, that $x_0 = 0$. By (3.3) it is enough to estimate the term $\int_D F(|u(x)-u_B|)\,dx$.

First we note that

$$
\begin{aligned}
\int_D F\big(|u(x)-u_B|\big)\,dx &\le \int_D F\Big(\fint_B |u(x)-u(y)|\,dy\Big)\,dx\\
&\le \int_B \fint_B F\big(|u(x)-u(y)|\big)\,dy\,dx\\
&\quad + \int_{D\setminus B}\fint_B F\big(|u(x)-u(y)|\big)\,dy\,dx.
\end{aligned}
\tag{5.4}
$$

The function F is increasing and by Theorem 1.2 a ball B is an F-Poincaré domain and hence

$$
\begin{aligned}
&\int_B \fint_B F\Big(\frac{1}{9\,\mathrm{dia}(D)}|u(x)-u(y)|\Big)\,dy\,dx\\
&\qquad \le \int_B \fint_B F\Big(\frac{1}{\mathrm{dia}(B)}|u(x)-u_B|\Big)\,dx\,dy\\
&\qquad \le c(n)\int_B F\big(|\nabla u(x)|\big)\,dx.
\end{aligned}
\tag{5.5}
$$

We estimate the last double integral in (5.4) in three parts:

$$
\begin{aligned}
&\int_{D\setminus B}\fint_B F\Big(\frac{1}{9\,\mathrm{dia}(D)}|u(x)-u(y)|\Big)\,dy\,dx\\
&\qquad\le \frac{1}{3}\int_{D\setminus B}\fint_B F\Big(\frac{1}{3\,\mathrm{dia}(D)}\Big|u(x)-u\Big(\frac{\ell}{2|x|}x\Big)\Big|\Big)\,dy\,dx\\
&\qquad\quad+\frac{1}{3}\int_{D\setminus B}\fint_B F\Big(\frac{1}{3\,\mathrm{dia}(D)}\Big|u\Big(\frac{\ell}{2|x|}x\Big)-u_B\Big|\Big)\,dy\,dx\\
&\qquad\quad+\frac{1}{3}\int_{D\setminus B}\fint_B F\Big(\frac{1}{3\,\mathrm{dia}(D)}|u(y)-u_B|\Big)\,dy\,dx,
\end{aligned}
\tag{5.6}
$$

where

$$
\begin{aligned}
&\int_{D\setminus B}\fint_B F\Big(\frac{1}{\mathrm{dia}(D)}|u(y)-u_B|\Big)\,dy\,dx\\
&\qquad\le c(n)\frac{|D\setminus B|}{|B|}\int_B F\big(|\nabla u(x)|\big)\,dx
\end{aligned}
\tag{5.7}
$$

by Theorem 1.2.

Since D is starshaped the first integral of the right hand side in (5.6) can be estimated by using spherical coordinates: for $\theta\in S^{n-1}(1)$ write $R(\theta)=|z|$ where z is the unique common point of ∂D and the ray $t\theta$, $t\ge 0$. Thus

$$
\begin{aligned}
&\int_{D\setminus B}\fint_B F\Big(\frac{1}{\mathrm{dia}(D)}\Big|u(x)-u\Big(\frac{\ell}{2|x|}x\Big)\Big|\Big)\,dy\,dx\\
&\qquad=\int\limits_{S^{n-1}(1)}\int\limits_{\ell/2}^{R(\theta)} F\Big(\frac{1}{\mathrm{dia}(D)}\Big|u(r,\theta)-u\Big(\frac{\ell}{2},\theta\Big)\Big|\Big)r^{n-1}\,dr\,dm_{n-1}(\theta).
\end{aligned}
$$

Applying the inequalities $\ell/2\le r\le R(\theta)\le L$ and $\ell/2=\mathrm{dia}(B)\le\alpha\le r$ we obtain

$$
\begin{aligned}
&\int_{\ell/2}^{R(\theta)} F\Big(\frac{1}{\mathrm{dia}(D)}\Big|u(r,\theta)-u\Big(\frac{\ell}{2},\theta\Big)\Big|\Big)r^{n-1}\,dr\\
&\qquad\le\int_{\ell/2}^{R(\theta)} F\Big(\frac{1}{|R(\theta)-\ell/2|}\int_{\ell/2}^{r}|u_\alpha(\alpha,\theta)|\,d\alpha\Big)r^{n-1}\,dr\\
&\qquad\le\int_{\ell/2}^{R(\theta)} F\Big(\frac{1}{|R(\theta)-\ell/2|}\int_{\ell/2}^{R(\theta)}|u_\alpha(\alpha,\theta)|\,d\alpha\Big)r^{n-1}\,dr\\
&\qquad\le\int_{\ell/2}^{R(\theta)} \frac{1}{|R(\theta)-\ell/2|}\int_{\ell/2}^{R(\theta)}F\big(|u_\alpha(\alpha,\theta)|\big)\,d\alpha r^{n-1}\,dr\\
&\qquad\le\int_{\ell/2}^{R(\theta)} \frac{1}{\ell/2}\int_{\ell/2}^{L}F\big(|u_\alpha(\alpha,\theta)|\big)\,d\alpha r^{n-1}\,dr\\
&\qquad\le\frac{2}{n}\frac{L^n}{\ell}\int_{\ell/2}^{R(\theta)}F\big(|\nabla u(\alpha,\theta)|\big)\frac{\alpha^{n-1}}{\alpha^{n-1}}\,d\alpha\\
&\qquad\le\frac{1}{n}\Big(\frac{2L}{\ell}\Big)^n\int_{\ell/2}^{R(\theta)}F\big(|\nabla u(\alpha,\theta)|\big)\alpha^{n-1}\,d\alpha.
\end{aligned}
$$

Hence

$$\int_{D\setminus B} \fint_B F\Big(\frac{1}{\operatorname{dia}(D)}\Big|u(x)-u\Big(\frac{\ell}{2|x|}x\Big)\Big|\Big)\,dy\,dx \\ \leq c(n)\Big(\frac{L}{\ell}\Big)^n \int_{D\setminus B} F\big(|\nabla u(x)|\big)\,dx. \tag{5.8}$$

In order to estimate the second integral of the right hand side of (5.6) we need Lemma 5.2. Changing the variables and using Lemma 5.2 and Theorem 1.2 we obtain

$$\begin{aligned}
&\int_{D\setminus B} \fint_B F\Big(\frac{1}{\operatorname{dia}(D)}\Big|u\Big(\frac{\ell}{2|x|}x\Big)-u_B\Big|\Big)\,dy\,dx \\
&\leq \int_{\ell/2}^{L} \int_{S^{n-1}(r)} F\Big(\frac{1}{2\ell}\Big|u\Big(\frac{\ell}{2|x|}x\Big)-u_B\Big|\Big)\,dm_{n-1}(x)\,dr \\
&= \int_{\ell/2}^{L} \int_{S^{n-1}(\ell/2)} F\Big(\frac{1}{2\ell}|u(z)-u_B|\Big)\frac{r^{n-1}}{(\ell/2)^{n-1}}\,dm_{n-1}(z)\,dr \\
&\leq c(n)\Big(\frac{L}{\ell}\Big)^{n-1} L \int_{S^{n-1}(\ell/2)} F\Big(\frac{1}{2\ell}|u(z)-u_B|\Big)\,dm_{n-1}(z) \\
&\leq c(n)\Big(\frac{L}{\ell}\Big)^{n} \int_D F\big(|\nabla u(x)|\big)\,dx.
\end{aligned} \tag{5.9}$$

Estimates (5.4)–(5.9) and (3.3), where $G = D$, together yield the inequality (1.1) with $b = \frac{1}{18}$.

6. **Further remarks.** We need an additional assumption of F to get $b = 1$ in inequality (1.1) for more general domains than convex domains. In this case, a variation of inequality (1.1) is the following one which was studied by Bhattacharya and Leonetti

$$\int_D F\Big(\frac{|u(x)-u_D|}{\operatorname{dia}(D)}\Big)\,dx \leq K_F \int_D F\big(|\nabla u(x)|\big)\,dx, \tag{6.1}$$

where D is a bounded domain in R^n, u is a function from an appropriate Sobolev space, $F\colon [0,\infty) \longrightarrow [0,\infty)$ is a convex, continuous function satisfying the Δ_2-condition, and $F(0) = 0$. Here constant K_F depends at most on F and D. By the Δ_2-condition we mean that there is a constant τ_F such that $F(2x) \leq \tau_F F(x)$ for all $x > 0$.

Then Theorems 4.1 and 5.1 read as

THEOREM 6.2. *An (α,β)-John domain in R^n satisfies the inequality (6.1) with $K_F = c(n)(\frac{\beta}{\alpha})^{n+1}\tau_F^{\eta}$; here $\eta = \eta(\frac{\alpha}{\beta}) < 0$.*

THEOREM 6.3. *A starshaped domain in R^n satisfies inequality (6.1) with a constant $K_F = \tau_F^{-5}K\big(n, d(x_0,\partial D), \max_{x\in\partial D} d(x,x_0)\big)$; here K is a constant from Theorem 5.1.*

The proofs for Theorems 6.2 and 6.3 are essentially the same as the proofs for Theorems 4.1 and 5.1.

6.4. REMARK. Let $F_i\colon[0,\infty)\to[0,\infty)$, $i=1,2$, be continuous functions with constants c_1 and c_2 such that the inequalities $c_1F_1(x)\le F_2(x)\le c_2F_1(x)$ hold for all $x\in[0,\infty)$. If F_1 is a convex function and $F_1(0)=0$, then D is an F_2-Poincaré domain whenever D is an F_1-Poincaré domain in the sense of (1.1). Further if F_1 satisfies the Δ_2-condition and D satisfies (6.1) with F_1, then D is an F_2-Poincaré domain in the sense of (6.1).

ACKNOWLEDGMENTS. I wish to thank Peter Lindqvist for bringing this problem to my attention and Jouni Luukkainen for carefully reading the manuscript.

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Canad. Math. Bull. Vol. **39** (4), 1996 pp. 448–459

ON BODIES ASSOCIATED WITH A GIVEN CONVEX BODY

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ABSTRACT. Let $d \geq 2$, and $K \subset \mathbb{R}^d$ be a convex body with $0 \in \operatorname{int} K$. We consider the intersection body IK, the cross-section body CK and the projection body ΠK of K, which satisfy $IK \subset CK \subset \Pi K$. We prove that $[\mathrm{bd}(IK)] \cap [\mathrm{bd}(CK)] \neq \emptyset$ (a joint observation with R. J. Gardner), while for $d \geq 3$ the relation $[CK] \subset \operatorname{int}(\Pi K)$ holds for K in a dense open set of convex bodies, in the Hausdorff metric. If $IK = c \cdot CK$ for some constant $c > 0$, then K is centred, and if both IK and CK are centred balls, then K is a centred ball. If the chordal symmetral and the difference body of K are constant multiples of each other, then K is centred; if both are centred balls, then K is a centred ball. For $d \geq 3$ we determine the minimal number of facets, and estimate the minimal number of vertices, of a convex d-polytope P having no plane shadow boundary with respect to parallel illumination (this property is related to the inclusion $[CP] \subset \operatorname{int}(\Pi P)$).

1. **Introduction.** For $d \geq 2$, let $K \subset \mathbb{R}^d$ be a convex body. We recall the definition of the *intersection body IK* of K, of the *cross-section body CK* of K, and of the *projection body* ΠK of K.

The body IK, for K a convex body with the origin 0 as an interior point (introduced by Lutwak [Lu], *cf.* also [Ga], Definition 8.1.1), is the star body with (necessarily continuous) radial function $V_{d-1}(K \cap u^{\perp})$ for $u \in S^{d-1}$, where $u^{\perp}$ is the linear $(d-1)$-subspace orthogonal to the unit vector u, and V_{d-1} means $(d-1)$-dimensional Lebesgue measure. In a sense, the history of intersection bodies started with the paper [Bu], where it is proved that, if K is a convex body in $\mathbb{R}^d$ centred at the origin, then IK also is a convex body with centre 0. The term *intersection body* appears later, namely in [Lu]. Intersection bodies are important for considering dual mixed volumes and questions like the famous Busemann-Petty problem (see [Lu] and [Ga], chapter 8). For example, any sufficiently smooth and strictly convex body in $\mathbb{R}^3$ with centre 0 is the intersection body of a star body, where the convex body K in the definition of IK above is replaced by a star-shaped body.

The body CK (introduced by [Ma 92], *cf.* also [Ga], Definition 8.3.1) is the star body with (necessarily continuous) radial function $\max_{\lambda \in \mathbb{R}} V_{d-1}\big(K \cap (u^{\perp} + \lambda u)\big)$ for $u \in S^{d-1}$. It was shown in [MM], Part II, that the cross-section body CT of a regular tetrahedron $T \subset \mathbb{R}^3$ is a cube, and an interesting open problem is the question whether cross-section bodies are convex, posed in [Ma 94], p. 279.

The research of the first author was (partially) supported by Hungarian National Foundation for Scientific Research, grant no. T-014285, and by Deutsche Forschungsgemeinschaft

Received by the editors June 8, 1995.

AMS subject classification: Primary: 52A20; Secondary: 52B11.

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The body ΠK, going back to Minkowski, is the convex body with support function $V_{d-1}(K \mid u^{\perp})$, where $K \mid u^{\perp}$ is the orthogonal projection of K to $u^{\perp}$ (*cf.* [BF], Section 30, p. 50, where it is also shown that $V_{d-1}(K \mid u^{\perp})$ is in fact the support function of a centred convex body; see also [Ga], Definition 4.1.1). The set of projection bodies is equal to the set of d-dimensional *zonoids* centred at the origin, *i.e.*, every such zonoid is the projection body of a certain class of convex bodies, and each projection body is such a zonoid. Zonoids are limits of zonotopes with respect to the Hausdorff metric, and zonotopes are vector sums of finitely many line segments. The literature on zonoids is widely covered by the surveys [SW], [GW], and [Ma 94]. For example, one sees easily that for T a regular tetrahedron in 3-space, ΠT is a rhombic dodecahedron, and for special convex double cones $C \subset \mathbb{R}^3$ with axial symmetry, ΠC is a spindle-shaped body formed by revolving a cosine curve around the corresponding coordinate axis.

We have evidently $IK \subset CK$ for $0 \in \operatorname{int} K$, and $CK \subset \Pi K$ by [Pe 52], p. 60, and [Ma 89], the latter stating, equivalently, that the radial function of CK is at most the reciprocal of the distance function of ΠK, *i.e.*, is at most the radial function of ΠK; *cf.* also [Ga], Theorem 8.3.3. For $d = 2$, one easily sees $CK = \Pi K =$ the convex body obtained from $K + (-K)$ by rotating it through $\frac{\pi}{2}$ about the origin, see [MM], Section 1, and [Ga], Theorems 4.1.4 and 8.3.5.

In this paper we prove some theorems related to the bodies IK, CK and ΠK.

Before Theorem 1, we recall some more definitions, *cf.* also [Fe], 2.10.1–2. For a metric space X and $m \geq 0$ the *m-dimensional Hausdorff measure H^m* is an outer measure defined on all subsets of X as follows: for $A \subset X$,

$$H^m(A) = \sup_{\delta>0}\Big(\inf\Big\{\sum_{i=1}^{\infty} \operatorname{diam}(A_i)^m \cdot \pi^{m/2}/\big(2^m\Gamma(1+\frac{m}{2})\big) \;\Big|\; A \subset \bigcup_{i=1}^{\infty} A_i \subset X,\ \forall i \operatorname{diam}(A_i) \leq \delta\Big\}\Big),$$

where diam means diameter. All closed subsets of X are H^m-measurable (see [Fe], pp. 54, 170). If m is a positive integer, one calls $A \subset X$, with $H^m(A) < \infty$, *(H^m, m)-rectifiable*, if

$$\forall \varepsilon > 0 \quad \exists A_\varepsilon \subset X, \quad H^m(A \setminus A_\varepsilon) < \varepsilon,$$

and A_ε is the image of a bounded subset of $\mathbb{R}^m$ by a Lipschitz map defined on this subset, *cf.* [Fe], pp. 251-252.

If X is a Euclidean space and A is a compact C^1 m-submanifold, then A is (H^m, m)-rectifiable, and $H^m(A)$ coincides with the differential geometric m-volume (Theorems 3.2.26 and 3.2.39 in [Fe]).

It should be noticed that a first version of Theorem 1 was jointly observed by R. J. Gardner and the second named author.

THEOREM 1. *For $d \geq 2$, let $K \subset \mathbb{R}^d$ be a convex body with $0 \in \operatorname{int} K$. Then we have $[\operatorname{bd}(IK)] \cap \operatorname{bd}(CK) \neq \emptyset$. More generally, even omitting the hypothesis $0 \in \operatorname{int} K$, there is a direction $u \in S^{d-1}$ such that $V_{d-1}(K \cap u^{\perp}) \geq V_{d-1}\big(K \cap (u^{\perp} + \lambda u)\big)$ holds*

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for each $\lambda \in \mathbb{R}$. *Moreover, for* $d \geq 3$ *the set of these* u*'s is not contained in any* H^{d-2}-*measurable,* $(H^{d-2}, d-2)$-*rectifiable subset* A *of the unit sphere* S^{d-1}, *which satisfies* $H^{d-2}(A) < H^{d-2}(S^{d-2})$ *(in particular, in any compact* $C^1(d-2)$-*submanifold of* S^{d-1} *with* $(d-2)$-*volume less than that of* S^{d-2}*). The estimate is sharp, e.g. for* K *any non-centred ball.*

We note that for $d = 2$ the assertion of Theorem 1 follows from an analogous result of Hammer (*cf.* [Ha 51], [Ha 63], and [PC]) on 1-dimensional sections of convex bodies in $\mathbb{R}^d$, which will be discussed in Section 3.

THEOREM 2. *For* $d \geq 2$, *let* $K \subset \mathbb{R}^d$ *be a convex body with* $0 \in \operatorname{int} K$. *If for some constant* $c > 0$ *we have* $IK = c \cdot CK$, *then* K *is centred. More generally, even omitting the hypothesis* $0 \in \operatorname{int} K$, *if for every* $u \in S^{d-1}$ *we have* $V_{d-1}(K \cap u^{\perp}) = c \cdot \max_{\lambda \in \mathbb{R}} V_{d-1}\big(K \cap (u^{\perp} + \lambda u)\big)$, *where* $c \geq 0$, *then the same conclusion holds.*

The following theorem is a characterization of balls.

THEOREM 3. *For* $d \geq 2$, *let* $K \subset \mathbb{R}^d$ *be a convex body with* $0 \in \operatorname{int} K$. *If both* IK *and* CK *are centred balls, then* K *is a centred ball. More generally, even omitting the hypothesis* $0 \in \operatorname{int} K$, *if both* $V_{d-1}(K \cap u^{\perp})$ *and* $\max_{\lambda \in \mathbb{R}} V_{d-1}\big(K \cap (u^{\perp} + \lambda u)\big)$ *are constant for* $u \in S^{d-1}$, *then the same conclusion holds.*

In Section 3, we will prove Proposition 1 that is the analogue of Theorems 2 and 3 for 1-dimensional sections and also includes a characterization of balls.

THEOREM 4. *For* $d \geq 3$, *let* $\mathfrak{K}^d$ *denote the set of convex bodies* $K \subset \mathbb{R}^d$, *endowed with the Hausdorff metric. Then* $\{K \in \mathfrak{K}^d \mid CK \subset \operatorname{int}(\Pi K)\}$ *is a dense open set in* $\mathfrak{K}^d$.

Related to the proof of Theorem 4 we will prove Proposition 2 (in Section 3), determining the minimal number of facets (vertices) of a convex polytope $P \subset \mathbb{R}^d$, $d \geq 3$ ($P \subset \mathbb{R}^3$) having no plane shadow boundary with respect to parallel illumination. For the question about the minimal number of vertices we give an estimate in $\mathbb{R}^d$, $d > 3$.

2. **Proofs of the Theorems.**

PROOF OF THEOREM 1. A. First we will show that, irrespective whether 0 is an interior, boundary or exterior point of K, any large 1-sphere S^1 of S^{d-1} (*i.e.*, the intersection of S^{d-1} with any linear 2-subspace) contains a point u satisfying $V_{d-1}(K \cap u^{\perp}) = \max_{\lambda \in \mathbb{R}} V_{d-1}\big(K \cap (u^{\perp} + \lambda u)\big)$. Letting $f_u(\lambda) = V_{d-1}\big(K \cap (u^{\perp} + \lambda u)\big)$, we have to prove that, for some $u \in S^1$, $f_u(\lambda)$ attains its maximum at $\lambda = 0$.

By the Brunn-Minkowski theorem $f_u(\lambda)^{1/(d-1)}$ is concave for $\lambda \in [-h_K(-u), h_K(u)] := [a, b]$, where h_K is the support function of K. Thus, $\big\{\lambda \in [a,b] \mid f_u(\lambda) = \max\{f_u(\mu) \mid \mu \in [a,b]\}\big\}$ is a non-empty closed interval, $I(u)$, say. Evidently, $I(-u) = -I(u)$. We consider the linear hull $\operatorname{lin} S^1$ of S^1, and the orthogonal projection $K \mid \operatorname{lin} S^1$ of K to $\operatorname{lin} S^1$. Now we distinguish three cases: 0 is an interior, boundary, or exterior point of $K \mid \operatorname{lin} S^1$, relative to $\operatorname{lin} S^1$.

First we investigate the case $0 \in \operatorname{int}(K \mid \operatorname{lin} S^1)$. Let $u \in S^1$. For $0 \in I(u)$ we are done. Thus, we may suppose that for all $u \in S^1$ either $I(u) \subset (0, \infty)$, or $I(u) \subset (-\infty, 0)$. Hence $S^1 = S^1_+ \cup S^1_-$, where $S^1_+ = \{u \in S^1 \mid I(u) \subset (0, \infty)\}$, $\quad S^1_- = \{u \in S^1 \mid I(u) \subset (-\infty, 0)\}$. Let u_0 be a common accumulation point of S^1_+ and S^1_-. If *e.g.* $I(u_0) \subset (0, \infty)$, then for some $\lambda_0 \in \bigl(0, h_K(u_0)\bigr)$ we have $V_{d-1}(K \cap u_0^\perp) < V_{d-1}\bigl(K \cap (u_0^\perp + \lambda_0 u_0)\bigr)$. Then, for $u \in S^1_-$ and sufficiently near to u_0 we have $V_{d-1}(K \cap u^\perp) < V_{d-1}\bigl(K \cap (u^\perp + \lambda u)\bigr)$, where $u^\perp + \lambda u$ is a hyperplane obtained from $u_0^\perp + \lambda_0 u_0$ by rotation about some point of $(\operatorname{int} K) \cap (u_0^\perp + \lambda_0 u_0)$. Thus $u \in S^1_+$, a contradiction.

Secondly we investigate the case $0 \notin (K \mid \operatorname{lin} S^1)$. Then the set $\{u \in S^1 \mid K \cap u^\perp \neq \emptyset\}$ is the union of two disjoint closed opposite arcs of S^1; one of these we denote by $\widehat{u_1 u_2}$. Then $V_{d-1}(K \cap u^\perp)$ depends continuously on u, for u varying in $\widehat{u_1 u_2}$. We may suppose that for all $u \in \widehat{u_1 u_2}$ we have $V_{d-1}(K \cap u^\perp) < \max_{\lambda \in \mathbf{R}} V_{d-1}\bigl(K \cap (u^\perp + \lambda u)\bigr)$, and therefore either $I(u) \subset (0, \infty)$ or $I(u) \subset (-\infty, 0)$. Using this for $u = u_1, u_2$ we easily see that $I(u_1) \subset (0, \infty)$ and $I(u_2) \subset (-\infty, 0)$, or conversely. Now choosing a common accumulation point of $\{u \in \widehat{u_1 u_2} \mid I(u) \subset (0, \infty)\}$ and $\{u \in \widehat{u_1 u_2} \mid I(u) \subset (-\infty, 0)\}$, we obtain a contradiction like in the previous case.

Third we investigate the case $0 \in \operatorname{bd}(K \mid \operatorname{lin} S^1)$. Then there is a supporting hyperplane $u_0^\perp$ of K with $u_0 \in S^1$. Consider a fixed closed arc $\widehat{u_0(-u_0)}$ of S^1. Let the arc $\widehat{u_1 u_2}$ be the closure of $\{u \in \widehat{u_0(-u_0)} \mid (\operatorname{int} K) \cap u^\perp \neq \emptyset\}$. Again we may suppose that $V_{d-1}(K \cap u^\perp) < \max_{\lambda \in \mathbf{R}} V_{d-1}\bigl(K \cap (u^\perp + \lambda u)\bigr)$ holds for all $u \in \widehat{u_1 u_2}$, in particular for $u = u_1, u_2$. Thus $V_{d-1}(K \cap u_i^\perp) < V_{d-1}\bigl(K \cap (u_i^\perp + \lambda_i u_i)\bigr)$, for some hyperplane $u_i^\perp + \lambda_i u_i$, intersecting $\operatorname{int} K$. The function $V_{d-1}(K \cap u^\perp)$, $\quad u \in \widehat{u_1 u_2}$, is continuous for $u \in \operatorname{relint} \widehat{u_1 u_2}$, and is upper semicontinuous at u_1 and u_2. Now we choose some $v_1, v_2 \in \operatorname{relint} \widehat{u_1 u_2}$, close to u_1 and u_2, respectively. Then $V_{d-1}(K \cap v_i^\perp) < V_{d-1}\bigl(K \cap (v_i^\perp + \mu_i v_i)\bigr)$, where $v_i^\perp + \mu_i v_i$ is a hyperplane obtained from $u_i^\perp + \lambda_i u_i$ by rotation about some point of $(\operatorname{int} K) \cap (u_i^\perp + \lambda_i u_i)$. Then we easily see that $I(v_1) \subset (0, \infty)$ and $I(v_2) \subset (-\infty, 0)$, or conversely. Now considering the closed subarc $\widehat{v_1 v_2}$ of $\widehat{u_1 u_2}$ and choosing a common accumulation point of $\{u \in \widehat{v_1 v_2} \mid I(u) \subset (0, \infty)\}$ and $\{u \in \widehat{v_1 v_2} \mid I(u) \subset (-\infty, 0)\}$, we obtain a contradiction like above.

B. Now we show that, for $d \geq 3$ and irrespective whether 0 is an interior, boundary or exterior point of K, the set

$$B := \Bigl\{u \in S^{d-1} \mid V_{d-1}(K \cap u^\perp) = \max_{\lambda \in \mathbf{R}} V_{d-1}\bigl(K \cap (u^\perp + \lambda u)\bigr)\Bigr\}$$

is not contained in any set A like in the theorem. Suppose the contrary. By part A of this proof and by central symmetry, B intersects each large 1-sphere S^1 of S^{d-1} in at least two points. Hence this holds for A as well.

Now we recall a special case of an integral geometric kinematic formula on S^{d-1} from [Fe], Theorem 3.2.48. Let $0 \leq k \leq d-1$ be an integer, and let $A \subset S^{d-1}$ be H^k-measurable with $H^k(A) < \infty$, and let A be (H^k, k)-rectifiable. Then, letting S^{d-1-k} be a fixed large $(d-1-k)$-subsphere of S^{d-1}, we have

$$\mathrm{int}_{O(d)} H^0\bigl(A \cap g(S^{d-1-k})\bigr)\, d\Theta_n(g) = \frac{\Gamma(\frac{k+1}{2})\Gamma(\frac{d-k}{2})}{2\pi^{(d+1)/2}} H^k(A) \cdot H^{d-1-k}(S^{d-1-k}).$$

Here $H^0(X)$ is the 0-dimensional Hausdorff measure of X, that equals the cardinality of X if it is finite, and equals ∞ if the cardinality of X is infinite. Moreover, Θ_d is the invariant (Haar) measure on the orthogonal group $O(d)$ of $\mathbb{R}^d$, with $\Theta_d\big(O(d)\big) = 1$. Letting $k = d-2$, and using $H^0\big(A \cap g(S^1)\big) \geq 2$, both sides of the above equality are at least 2, and by a small calculation this yields $H^{d-2}(A) \geq H^{d-2}(S^{d-2})$. This contradicts our indirect assumption, and thus shows the statement of the theorem concerning A.

For K a ball with centre $a \neq 0$ we have $B = \{u \in S^{d-1} \mid \langle u, a\rangle = 0\}$, thus B is a large S^{d-2} of S^{d-1}, and therefore the estimate of $H^{d-2}(A)$ is sharp. ■

PROOF OF THEOREM 2. For each $u \in S^{d-1}$, let there hold the relation $V_{d-1}(K \cap u^{\perp}) = c \cdot \max_{\lambda \in \mathbb{R}} V_{d-1}\big(K \cap (u^{\perp} + \lambda u)\big)$. Then $c \leq 1$, and $c > 0$. Hence $V_{d-1}(K \cap u^{\perp}) > 0$ for each $u \in S^{d-1}$, which implies $0 \in K$.

For $0 \in \operatorname{bd} K$, the body K has a supporting hyperplane $u_0^{\perp}$ at 0, and $V_{d-1}(K \cap u_0^{\perp}) > 0$ implies that $K \cap u_0^{\perp}$ is a $(d-1)$-face of K. Now let us move u on a large 1-sphere S^1 of S^{d-1}, containing u_0. Then for $u \longrightarrow u_0$ the sum of the two one-sided limits of $V_{d-1}(K \cap u^{\perp})$ is $V_{d-1}(K \cap u_0^{\perp})$; hence u_0 is a discontinuity point of $V_{d-1}(K \cap u^{\perp})$. Since $\max_{\lambda \in \mathbb{R}} V_{d-1}\big(K \cap (u^{\perp} + \lambda u)\big)$ is a continuous function of u, this is a contradiction, implying $0 \in \operatorname{int} K$. Therefore from $IK = c \cdot CK, c \in [0, 1]$, we have by Theorem 1 that $c = 1$ and $IK = CK$. In [MMÓ] it was shown that for $0 \in \operatorname{int} K$ the coincidence of IK and CK implies that K is centred. This implies the result. ■

PROOF OF THEOREM 3. Let both $V_{d-1}(K \cap u^{\perp})$ and $\max_{\lambda \in \mathbb{R}} V_{d-1}\big(K \cap (u^{\perp} + \lambda u)\big)$ be constant for $u \in S^{d-1}$. Then by Theorem 2 we have that K is centred. Recall now the section theorem of Funk: a centred convex body K is uniquely determined by IK, *cf.* [Fu], p. 287, [Pe 52], Corollary 1.31, [LP], p. 1144, [Pe 61], Theorem 4.2, and [Fa], Theorem 2; see also [Ga], Corollary 7.2.7. Applying this to K and using that IK is a centred ball, we obtain that K is a centred ball. ■

To prove Theorem 4, we need some preparations. First we recall three definitions, for which we also refer to [BES], [Sh], [GH], and [Pe 83].

DEFINITION 1. Let $K \subset \mathbb{R}^d$ be a convex body, s a direction, and $x \in \mathbb{R}^d \setminus K$ a point. The *shadow boundary* $\operatorname{Sb}_s K$ ($\operatorname{Sb}_x K$) of K with respect to parallel illumination from the direction s (central illumination from the point x) is the intersection of K and the union of all its supporting lines parallel to s (passing through x). ■

DEFINITION 2. A shadow boundary $\operatorname{Sb}_s K$ ($\operatorname{Sb}_x K$) is said to be a *plane shadow boundary* if it lies in a hyperplane. ■

DEFINITION 3. A shadow boundary $\operatorname{Sb}_s K$ ($\operatorname{Sb}_x K$) is called a *generalized plane shadow boundary* provided there exists a hyperplane H not parallel to s (not containing x and intersecting each ray $xy, y \in K$, transversally) such that $H \cap \operatorname{bd}\big(K + l(s)\big) \subset \operatorname{Sb}_s K$, where $l(s)$ is a line of direction s passing through 0 ($H \cap \operatorname{bd} C(x, K) \subset \operatorname{Sb}_x K$, where $C(x, K)$ is the minimal infinite cone with apex x that contains K). ■

One easily sees that plane shadow boundaries can be equivalently defined by replacing the inclusion signs in Definition 3 by equalities.

Now we recall a result from [Pe 52], p. 60, and [Ma 89], *cf.* the Theorem in the latter, in the equivalent reformulation mentioned in our Section 1, fifth paragraph. For $d \geq 3$, let $K \subset \mathbb{R}^d$ be a convex body, and let $u \in S^{d-1}$. Then the radial function of CK at u is at most that of ΠK at u, with equality if and only if the following condition is satisfied: There is a direction s not orthogonal to u and some $\lambda \in \mathbb{R}$ such that for the shadow boundary $\mathrm{Sb}_s K$ of K with respect to parallel illumination from the direction s we have $(u^\perp + \lambda u) \cap \mathrm{bd}\left(K + l(s)\right) \subset \mathrm{Sb}_s K$ (again, $l(s)$ is a line of direction s passing through 0).

An easy consequence of this result is the following

LEMMA 1. *For $d \geq 3$, a convex body $K \subset \mathbb{R}^d$ satisfies $CK \subset \mathrm{int}(\Pi K)$ if and only if it does not have a generalized plane shadow boundary $\mathrm{Sb}_s K$ with respect to parallel illumination from any direction s.*

PROOF. By the above result, the following four statements are equivalent.

(1) $CK \not\subset \mathrm{int}(\Pi K)$.

(2) There is some $u \in S^{d-1}$ such that the radial functions of CK and ΠK are equal at u.

(3) There is some $u \in S^{d-1}$ and some direction s (where u and s are not orthogonal) such that there exists some $\lambda \in \mathbb{R}$ with $(u^\perp + \lambda u) \cap \mathrm{bd}\big(K + l(s)\big) \subset \mathrm{Sb}_s K$.

(4) There is some direction s such that $\mathrm{Sb}_s K$ is a generalized plane shadow boundary. ∎

Now we turn to the space $\mathfrak{K}^d$ of all convex bodies $K \subset \mathbb{R}^d$, endowed with the Hausdorff metric. This is a locally compact metric space, hence a Baire space, *i.e.*, a countable union of nowhere dense sets (otherwise: a set of first category) cannot have an interior point. One says that *most convex bodies in $\mathbb{R}^d$ have some property* if the set of convex bodies in $\mathbb{R}^d$ not having this property is of first category in $\mathfrak{K}^d$. For this concept and results concerning it we refer to the surveys of Gruber [Gr 85], [Gr 93] and Zamfirescu [Za 91a].

Now we recall two results on most convex bodies. Namely, most convex bodies $K \subset \mathbb{R}^d$ are strictly convex and smooth, by [Kl], Theorems 2.2 and 2.3 (which actually prove a version of this for Banach spaces) and [Gr 77], Satz 1; *cf.* also [Gr 85], pp. 163-164, [Za 91a], Theorem 1, and [Gr 93], Theorem 5. Also most convex bodies $K \subset \mathbb{R}^d$ have no plane shadow boundaries $\mathrm{Sb}_s K$ or $\mathrm{Sb}_x K$, for s any direction and $x \in \mathbb{R}^d \setminus K$ any point, by [Za 91b], Theorem 1; *cf.* also [Za 91a], Theorem 12.

An easy consequence of these results is the following lemma, whose proof is probably quite similar to the proof of the case $\dim L = d - 1$ of Theorem 31, announced without proof in [Gr 93].

LEMMA 2. *Most convex bodies $K \subset \mathbb{R}^d$ do not have any generalized plane shadow boundaries $\mathrm{Sb}_s K$ or $\mathrm{Sb}_x K$, for any direction s and any point $x \in \mathbb{R}^d \setminus K$.*

PROOF. By the above recalled two results, most convex bodies $K \subset \mathbb{R}^d$ both are strictly convex and have no plane shadow boundaries $\mathrm{Sb}_s K$ or $\mathrm{Sb}_x K$, with s, x as above. However, a convex body $K \subset \mathbb{R}^d$ with these two properties cannot have a generalized plane shadow boundary $\mathrm{Sb}_s K$ or $\mathrm{Sb}_x K$ either. This is easily seen by using the equivalent definition of plane shadow boundary after Definition 3, and that K contains exactly one point of any generator of $K + l(s)$, or of $C(x, K)$, respectively. ∎

PROOF OF THEOREM 4. Let $\mathfrak{K}' = \{K \in \mathfrak{K}^d \mid CK \subset \operatorname{int}(\Pi K)\}$. Both CK and ΠK are monotonous functions of K with respect to inclusion, of homogeneity $d-1$, and are translation invariant, that readily implies their continuity. (For ΠK we use the topology of Hausdorff-distance, for CK that of uniform convergence of the radial functions.) Since also

$$\mathfrak{K}' = \{K \in \mathfrak{K}^d \mid \exists \varepsilon > 0, \quad CK \subset (1-\varepsilon)\Pi K\},$$

we have by the above continuity properties that $\mathfrak{K}'$ is open.

By Lemma 1 we have $CK \subset \operatorname{int}(\Pi K)$ provided K does not have any generalized plane shadow boundary $\mathrm{Sb}_s K$, for any direction s. Hence by Lemma 2 most convex bodies $K \subset \mathbf{R}^d$ satisfy $CK \subset \operatorname{int}(\Pi K)$. In particular, this is satisfied for K in a dense set of $\mathfrak{K}^d$. Hence the open set $\mathfrak{K}' \subset \mathfrak{K}^d$ also is dense, and the proof is finished. ∎

3. **Proofs of the propositions.** 1. Replacing $(d-1)$-dimensional sections and projections by 1-dimensional sections and projections, one gets the following natural comparison. Let $d \geq 2$ and let $K \subset \mathbf{R}^d$ be a convex body. For $0 \in \operatorname{int} K$ let $\tilde{\Delta} K$ be the star-body whose radial function is given by $V_1(K \cap l_u)/2$, where V_1 is 1-dimensional Lebesgue measure and l_u is the linear 1-subspace with $u \in S^{d-1}$ as its direction vector. The body $\tilde{\Delta} K$ is said to be the *chordal symmetral* of K, see [Ga], Definition 5.1.3, and it is clear that $2\tilde{\Delta} K$ is the analogue (for 1-dimensional sections) of the intersection body IK. On the other hand, the *difference body* $DK = K + (-K)$ (see, *e.g.*, [Ga], Section 3.2) is the analogue of the cross-section body CK, and, at the same time, of the projection body ΠK, depending on the use of the radial function and of the support function of DK, respectively. We have evidently $2\tilde{\Delta} K \subset DK$ for $0 \in \operatorname{int} K$.

It turns out that the statements analogous to our first three theorems hold true, except the statement about A in Theorem 1 (here, for any non-centred ball, u is unique up to sign). Namely, Hammer (see [Ha 63], Theorem 3.1, and [Ha 51], Theorem 1, and also [PC], proof of Theorem 4) has proved that each $x \in \mathbf{R}^d$ belongs to a diametrical chord of K, which is an analogue of our Theorem 1, without the statement about A. (A chord of K is said to be *diametrical* if it has maximal length among all chords of K parallel to it.) This shows $[\operatorname{bd}(2\tilde{\Delta} K)] \cap \operatorname{bd}(DK) \neq \emptyset$, and therefore we obtain the following analogue of Theorems 2 and 3, also containing a characterization of balls.

PROPOSITION 1. *For $d \geq 2$, let $K \subset \mathbf{R}^d$ be a convex body with $0 \in \operatorname{int} K$. If for some constant $c > 0$ we have $2\tilde{\Delta} K = c \cdot DK$, then K is centred. In particular, if $2\tilde{\Delta} K$ and DK are centred balls, then K is a centred ball. More generally, even omitting the hypothesis $0 \in \operatorname{int} K$, if $V_1(K \cap l_u)$ is proportional to the radial function of DK, then K is centred. In particular, if these two functions are constant, then K is a centred ball.*

PROOF. We proceed analogously as with respect to Theorems 2 and 3. So we only indicate the differences in the proof. If $V_1(K \cap l_u)$ is positive and continuous for $u \in S^{d-1}$, then $0 \in \operatorname{int} K$. Namely, if $0 \in \operatorname{bd} K$, u_0 is an outer normal of K at 0, and $u_1 \in u_0^{\perp} \cap S^{d-1}$, then for $u \longrightarrow u_1$ (u varying in the large 1-sphere S^1 of S^{d-1} containing u_0 and u_1) the sum of the two one-sided limits of $V_1(K \cap l_u)$ is $V_1(K \cap l_{u_1})$. Then by $[\operatorname{bd}(2\tilde{\Delta}K)] \cap \operatorname{bd}(DK) \neq \emptyset$ the proportionality of $V_1(K \cap l_u)$ and the radial function of DK implies centredness of K, *cf.* [Ha 54], Theorem 1, and also [PC], Theorem 4. The particular cases follow immediately. ■

It is obvious that, regarding 1-dimensional sections and projections, there is no analogue of Theorem 4.

2. Let $d \geq 3$. By Theorem 4 and denseness of convex polytopes in $\mathfrak{K}^d$ we see that there are convex polytopes $P \subset \mathbb{R}^d$ such that $CP \subset \operatorname{int}(\Pi P)$. (We always will consider non-degenerate convex polytopes, *i.e.*, $\operatorname{int} P \neq \emptyset$.) Thus, by Lemma 1 there are convex polytopes $P \subset \mathbb{R}^d$ that do not have a generalized plane shadow boundary $\mathrm{Sb}_s P$ with respect to parallel illumination from any direction s. We ask for the minimal complexity (for example, number of vertices, or facets) of a convex polytope $P \subset \mathbb{R}^d$ having no generalized plane shadow boundary, or no plane shadow boundary $\mathrm{Sb}_s P$, with respect to parallel illumination from any direction s.

Some of these questions for plane shadow boundaries are answered by

PROPOSITION 2. *Let $d \geq 3$ be an integer. The minimal number of facets (vertices) of a non-degenerate convex polytope $P \subset \mathbb{R}^d$, having no plane shadow boundary $\mathrm{Sb}_s P$ with respect to parallel illumination from any direction s, is $d+2$ (at most $2d$, with equality for $d=3$).*

PROOF. A. First we deal with the number of facets. Since a simplex has plane shadow boundaries $\mathrm{Sb}_s P$, it suffices to give an example of a (non-degenerate) convex polytope P with $d+2$ facets having no plane shadow boundary $\mathrm{Sb}_s P$ with respect to parallel illumination. Such a polytope will be a prism P over a $(d-1)$-simplex, whose bases and lateral facets will be denoted by B_1, B_2 and $L_1, \ldots, L_d$, respectively. Namely, if the direction s is parallel to the generator or to a basis of P, then $\mathrm{Sb}_s P$ contains some facets of P, and hence $\mathrm{Sb}_s P$ is not planar. Otherwise, a unit vector of direction s is of the form $x+y$, $x \neq 0$ parallel to a base, $y \neq 0$ parallel to the generator of P. Like above, we may suppose that $\mathrm{Sb}_s P$ does not contain any facet of P. Then the illuminated facets are one of the bases (B_1, say) and a non-empty proper subset ($\{L_1, \ldots, L_k\}$, say) of the lateral facets, where L_i is a prism over a facet l_i of B_1. Then $\mathrm{Sb}_s P \supset (B_1 \cap L_{k+1}) \cup \cdots \cup (B_1 \cap L_d) \cup (B_2 \cap L_1) \cup \cdots \cup (B_2 \cap L_k)$, that is the union of $d-k$ facets of B_1 and k facets of B_2. Since $\max\{d-k, k\} \geq 2$ and $\min\{d-k, k\} \geq 1$, the shadow boundary $\mathrm{Sb}_s P$ contains all vertices of B_1 and all but one vertices of B_2, or conversely. Hence $\mathrm{Sb}_s P$ is not planar.

B. Secondly we deal with the number of vertices and begin by giving an example of a convex polytope $P \subset \mathbb{R}^d$, $d \geq 3$, with $2d$ vertices having no plane shadow boundary $\mathrm{Sb}_s P$ with respect to parallel illumination.

First, let P_0 denote the cross-polytope with vertices $\pm e_i$ (e_i is the i-th basic unit vector). We claim that its only plane shadow boundaries $\mathrm{Sb}_s P_0$ with respect to parallel illumination are the intersections of $\operatorname{bd} P_0$ with the d coordinate hyperplanes. Let $\mathrm{Sb}_s P_0$ be a plane shadow boundary of P_0 with respect to parallel illumination from some direction s. (The following considerations in this paragraph will be valid for any convex d-polytope rather than P_0.) Choose a new orthogonal coordinate system $y_1, \dots, y_d$, where y_d has direction s. Let π denote the projection of $\mathbb{R}^d$ to the $(y_1, \dots, y_{d-1})$-hyperplane, and let $P_0' = \pi(P_0)$. Then P_0' is a convex $(d-1)$-polytope, and $\operatorname{Sb}_s P_0$ is the graph of the restriction of an affine function $f\colon \mathbb{R}^{d-1} \longrightarrow \mathbb{R}$ to $\operatorname{relbd} P_0'$ ($\mathbb{R}^{d-1}$ denoting the $(y_1, \dots, y_{d-1})$-hyperplane, and $\mathbb{R}$ denoting the y_d-axis). Let us denote the facets of P_0' by F_i'. Then $F_i = P_0 \cap \pi^{-1}(F_i') \subset \operatorname{Sb}_s P_0$ are non-degenerate affine images of F_i', and by $F_i' = P_0' \cap \operatorname{aff} F_i'$ also the sets $F_i = P_0 \cap \pi^{-1}(\operatorname{aff} F_i')$ are faces of P_0, of dimension $d-2$. Conversely, each $(d-2)$-face F of P_0, contained in $\operatorname{Sb}_s P_0$, also is a $(d-2)$-face of $\operatorname{conv}\big(\operatorname{Sb}_s(P_0)\big)$, which is affinely equivalent under the map π to P_0'. Thus, each such F is of the above form F_i. Each $(d-3)$-face of P_0' is a common face of exactly two facets F_i' of P_0'. Therefore, for each $(d-2)$-face of P_0, of the form F_i, and each $(d-3)$-face G of F_i there is another $(d-2)$-face of P_0, of the form F_{i_1}, also having G as a $(d-3)$-face.

From now on we will exploit the combinatorial and affine structure of P_0. The $(d-2)$-faces ($(d-3)$-faces) of P_0 are given by equations $\sum_{i \neq k} \varepsilon_i x_i = 1$, $x_k = 0$ ($\sum_{i \neq k,l} \varepsilon_i x_i = 1, x_k = x_l = 0$, where $k \neq l$, here and later) with $\varepsilon_i \in \{-1, 1\}$. (Writing such sums we always will assume that the summands are non-negative, but will not repeat it in the following.) If two different $(d-2)$-faces F_1^{d-2}, F_2^{d-2} have a common $(d-3)$-face F^{d-3}, and if F^{d-3} is given by $\sum_{i \neq k,l} \varepsilon_i x_i = 1, \quad x_k = x_l = 0$, then each of the faces F_1^{d-2}, F_2^{d-2} is of the form

$$(1) \qquad \Big(\sum_{i \neq k,l} \varepsilon_i x_i\Big) \pm x_k = 1, \quad x_l = 0, \text{ or}$$

$$(2) \qquad \Big(\sum_{i \neq k,l} \varepsilon_i x_i\Big) \pm x_l = 1, \quad x_k = 0.$$

Thus, the sets F_1^{d-2}, F_2^{d-2} are two of four $(d-2)$-faces. They lie in the same facet of P_0 if and only if one of them is of the form (1) and the other is of the form (2). In this case, if moreover F_1^{d-2}, F_2^{d-2} are $(d-2)$-faces of P_0 contained in $\operatorname{Sb}_s P_0$, then, by planarity of $\operatorname{Sb}_s P_0$, we see that $\operatorname{Sb}_s P_0$ is a subset of this facet of P_0, F^{d-1}, say. Since $\operatorname{Sb}_s P_0$ also is a union of $(d-2)$-faces of P_0, of the above form F_i, and is affinely equivalent to $\operatorname{relbd} P_0'$, thus is homeomorphic to S^{d-2}, then $\operatorname{Sb}_s P_0 = \operatorname{relbd} F^{d-1}$. Therefore $P_0' = \operatorname{conv} \operatorname{relbd} P_0') = \operatorname{conv}\big(\pi(\operatorname{Sb}_s P_0)\big) = \pi\big(\operatorname{conv}(\operatorname{Sb}_s P_0)\big) = \pi(F^{d-1})$ is a $(d-1)$-simplex. For the opposite parallel facet $-F^{d-1}$ of P_0 we have $P_0' \supset \pi(-F^{d-1})$, and therefore $\pi(F^{d-1}) \supset \pi(-F^{d-1})$, a contradiction.

Thus, if a $(d-2)$-face F_1^{d-2} of P_0 is contained in $\operatorname{Sb}_s P_0$, then at any of its $(d-3)$-faces F^{d-3} it is neighbourly to a $(d-2)$-face F_2^{d-2} of P_0, contained in $\operatorname{Sb}_s P_0$, both F_1^{d-2}, F_2^{d-2} being of the form (1), or both being of the form (2) above. If F_1^{d-2} is given by $\sum_{i \neq k} \varepsilon_i x_i = 1, \quad x_k = 0$, then, for fixed F^{d-3}, the face F_2^{d-2} is unique and has the form

$$\Big(\sum_{i \neq k,l} \varepsilon_i x_i\Big) - \varepsilon_l x_l = 1, \quad x_k = 0,$$

where $l \in \{1, \dots, d\} \setminus \{k\}$ can be arbitrary, for arbitrary F^{d-3}. Repeating this consideration at most $d-1$ times, we see that each $(d-2)$-face of P_0 of the form $\sum_{i \neq k}(\pm x_i) = 1, \quad x_k = 0$, is contained in $\mathrm{Sb}_s P_0$. Hence $\mathrm{Sb}_s P_0$ contains, and therefore is equal to, the intersection of $\mathrm{bd}\, P_0$ with the k-th coordinate hyperplane.

Now let us perturb the vertices of P_0 sufficiently little, so that the combinatorial type of their convex hull P remains the same as that of P_0, and no $d+1$ vertices lie in a hyperplane. Supposing that P has a plane shadow boundary $\mathrm{Sb}_s P$ with respect to parallel illumination, we can repeat, for sufficiently small perturbations, the above considerations. In the first case observe that P_0' contains and is contained in some centred balls of fixed radii, and then we obtain a contradiction like above. In the second case the above existing d plane shadow boundaries (each containing $2d-2 \geq d+1$ vertices of P) will become non-planar, a contradiction once more.

Still we have to prove that convex polyhedra P in $\mathbb{R}^3$ with at most 5 vertices have plane shadow boundaries $\mathrm{Sb}_s P$ with respect to parallel illumination. For a tetrahedron this is evident. If P has 5 vertices, then it is either a double triangular pyramid, or a quadrangular pyramid. Then $\mathrm{Sb}_s P$ is planar for s parallel to the diagonal of P, or to the segment joining the apex of P with a relative interior point of its base, respectively. ■

4. **Concluding Remarks.** Remark 1. For $d = 2$, the convex body $K \subset \mathbb{R}^2$ has a cross-section body CK which is the convex body obtained from $K + (-K)$ by rotating it through $\frac{\pi}{2}$ about the origin, *cf.* Section 1. Thus, CK is a centred ball if and only if K is of constant width. For $d \geq 3$, no example of a convex body $K \subset \mathbb{R}^d$, different from a ball and having as CK a centred ball, is known. However, CK does not seem to give full information about K, since the informations obtained by u and $-u$ are the same, like also in the case of the difference body $K + (-K)$, or IK (this for $0 \in \mathrm{int}\, K$), or ΠK. (Roughly: we have 'half the information that is needed'.) In these latter cases, there is given complete information on the even parts of the support function, of the $(d-1)$-st power of the radial function (*cf.* [Ga], Theorem 8.1.3), and of the surface area measure (see [Ga], Theorem 3.3.2), respectively; however, no information is given about the odd parts. Therefore we can well imagine that there exists a convex body $K \subset \mathbb{R}^d$, $d \geq 3$, with CK a centred ball but not having itself a center of symmetry. For the history of this problem (the Bonnesen-Klee problem, also posed in [Ga], Problem 8.12) we refer to [Ga], Note 8.6.

Remark 2. Theorem 4 shows that, for $d \geq 3$, there is a number $c(d) < 1$ such that for some convex body $K \subset \mathbb{R}^d$ we have $CK \subset c(d) \cdot \Pi K$. What is the minimal number with this property? Surely it is positive, since by [MM], Part I, there is a constant $c'(d)$ such that for all convex bodies $K \subset \mathbb{R}^d$ we have $\Pi K \subset c'(d) \cdot CK$.

Remark 3. Let $1 < k < d-1$, and let $K \subset \mathbb{R}^d$ be a convex body. We ask if there is a linear k-subspace L_k such that

$$V_k(K \cap L_k) = \max\{V_k\big(K \cap (L_k + x)\big) \mid x \in \mathbb{R}^d\},$$

where V_k denotes k-dimensional Lebesgue measure. Moreover, if we consider the set of linear k-subspaces L_k of $\mathbb{R}^d$ with some natural metric, then is the set of L_k's with this maximum property large enough? For example, cannot it be included in a compact C^1 $(k-1)(d-k)$-submanifold, of measure less than in the case of a non-centred ball. (For K a non-centred ball, the set of these L_k's is homeomorphic to the set of linear $(k-1)$-subspaces of $\mathbb{R}^{d-1}$, which is a $(k-1)(d-k)$-manifold.) Probably some algebraic topology would be necessary to answer this question.

REMARK. (Added 11 December 1995): Meanwhile the authors proved the existence of L_k in Remark 3, and moreover that the set of all such L_k's cannot be included in a compact C^∞ $(k-1)(d-k)$-submanifold, of measure less than some positive constant depending on d and k, when the set of linear k-subspaces of $\mathbb{R}^d$ is given in a natural $O(d)$-invariant Riemann-metric, and measure is taken relative to this. Thus we have the analogues of Theorems 2 and 3 as well, for $1 < k < d-1$. For Theorem 4 there is a natural generalisation of the relation $CK \subset \operatorname{int}(\Pi K)$ for k-dimensional sections and projections: for K in a dense open set of $\mathfrak{K}^d$ there exists $c(K) < 1$ such that $\max\{V_k([K \cap (L_k+x)] \mid L_k') \; V_k(K \mid L_k') \mid L_k, L_k' \subset \mathbb{R}^d$ are linear k-subspaces, $x \in \mathbb{R}^d\} \leq c(K)$. (The set $A \mid L_k'$ is the orthogonal projection of the set A to L_k'.) This is connected with problems of illumination from a $(d-k-1)$-dimensional projective subspace of the infinite hyperplane.

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Canad. Math. Bull. Vol. 39 (4), 1996 pp. 460–467

SELF-ADJOINT FREDHOLM OPERATORS AND SPECTRAL FLOW

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ABSTRACT. We study the topology of the nontrivial component, $\mathcal{F}_*^{sa}$, of self-adjoint Fredholm operators on a separable Hilbert space. In particular, if $\{B_t\}$ is a path of such operators, we can associate to $\{B_t\}$ an integer, $\mathrm{sf}(\{B_t\})$, called the *spectral flow* of the path. This notion, due to M. Atiyah and G. Lusztig, assigns to the path $\{B_t\}$ the net number of eigenvalues (counted with multiplicities) which pass through 0 in the positive direction. There are difficulties in making this precise — the usual argument involves looking at the graph of the spectrum of the family (after a suitable perturbation) and then counting intersection numbers with $y = 0$.

We present a completely different approach using the functional calculus to obtain *continuous* paths of eigenprojections (at least locally) of the form $\chi_{[-a,a]}(B_t)$. The spectral flow is then defined as the dimension of the nonnegative eigenspace at the end of this path minus the dimension of the nonnegative eigenspace at the beginning. This leads to an easy proof that spectral flow is a well-defined homomorphism from the homotopy groupoid of $\mathcal{F}_*^{sa}$ onto Z. For the sake of completeness we also outline the seldom-mentioned proof that the restriction of spectral flow to $\pi_1(\mathcal{F}_*^{sa})$ is an isomorphism onto Z.

Introduction. Most workers in the field of (self-adjoint) operator algebras are quite conversant with the ideas and techniques of K-theory. While most (if not all) of us are aware of the Atiyah-Janich theorem [A] which states that the space, $\mathcal{F}$, of Fredholm operators is a classifying space for K^0 (that is, $K_0\big(C(X)\big) = K^0(X) \cong [X, \mathcal{F}]$) and that, by definition, $U(\infty) = \lim_n U(n)$ is a classifying space for K^1 (that is, $K_1\big(C(X)\big) = K^1(X) = [X, U(\infty)]$), some of us are less aware of the Atiyah-Singer result that the nontrivial component, $\mathcal{F}_*^{sa}$, of the self-adjoint Fredholm operators is also a classifying space for K^1. This is the central result of their paper on Index Theory for Skew-Adjoint Fredholm Operators [AS]. There are, of course, other realizations of a classifying space for K^1 but $\mathcal{F}_*^{sa}$ is central to the Index Theorem for families of self-adjoint elliptic operators. In particular, $\pi_1(\mathcal{F}_*^{sa}) = [S^1, \mathcal{F}_*^{sa}] \cong K^1(S^1) = \mathbf{Z}$ arises from the notion of *spectral flow* which is an important concept in index theory [W]. Heuristically, the spectral flow of a one-parameter family of self-adjoint Fredholm operators is just the net number of eigenvalues (counting multiplicities) which pass through zero in the positive direction from the start of the path to its end. The usual way of making this idea rigorous involves looking at the graph of the spectrum of the family after a suitable perturbation, and counting intersection numbers with $y = 0$ (taking into account multiplicities) [APS, BW]. This

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Received by the editors December 16, 1994; revised May 25, 1995.

AMS subject classification: Primary: 19K56; secondary: 46L80, 46M20.

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approach is appealing to topologists, but leaves some analysts feeling uneasy. We present here a completely different approach using the functional calculus for self-adjoint operators. The definition then involves counting dimensions of finite-rank projections and is slightly combinatorial in nature. The resulting map on paths is easily seen to be a well-defined, homotopy invariant homomorphism onto **Z**. This definition should appeal to specialists in operator algebras while also being accessible to topologists and geometers. For the sake of completeness we also provide the seldom-mentioned proof that the map is also one-to-one.

Spectral Flow. Let $\mathcal{H}$ be a separable Hilbert space and let $\mathcal{B} = \mathcal{B}(\mathcal{H})$, $\mathcal{K} = \mathcal{K}(\mathcal{H})$ denote the algebras of bounded and compact operators on $\mathcal{H}$, respectively. Let $\mathcal{Q} = \mathcal{Q}(\mathcal{H}) := \mathcal{B}/\mathcal{K}$ be the Calkin algebra and $\pi\colon \mathcal{B} \longrightarrow \mathcal{Q}$ the canonical homomorphism. Let $\mathcal{F} = \{T \in \mathcal{B} \mid \pi(T) \text{ is invertible in } \mathcal{Q}\}$ denote the set of Fredholm operators on $\mathcal{H}$ and $\mathcal{F}^{sa}$ the set of self-adjoint Fredholm operators on $\mathcal{H}$. The space $\mathcal{F}^{sa}$ has three components determined by the essential spectrum:

$$\begin{aligned}
\mathcal{F}^{sa}_{+} &:= \{T \in \mathcal{F}^{sa} \mid \pi(T) > 0\}\\
\mathcal{F}^{sa}_{-} &:= \{T \in \mathcal{F}^{sa} \mid \pi(T) < 0\}\\
\mathcal{F}^{sa}_{*} &:= \{T \in \mathcal{F}^{sa} \mid \operatorname{sp}(\pi(T)) \not\subseteq \mathbf{R}^{+} \text{ and } \operatorname{sp}(\pi(T)) \not\subseteq \mathbf{R}^{-}\}.
\end{aligned}$$

PROPOSITION 1. *$\mathcal{F}^{sa}_{+}$ and $\mathcal{F}^{sa}_{-}$ are contractible [AS].*

PROOF. For $(T, \lambda) \in \mathcal{F}^{sa}_{+} \times [0, 1]$ let $\phi(T, \lambda) = (1 - \lambda)T + \lambda 1$. Then, $\pi(\phi(T, \lambda)) = [(1 - \lambda)\pi(T) + \lambda 1] > 0$ so $\phi(T, \lambda) \in \mathcal{F}^{sa}_{+}$. Clearly, $\phi(T, 0) = T$ and $\phi(T, 1) = 1$. Similarly, $\mathcal{F}^{sa}_{-}$ is contractible to -1.

DISCUSSION. We give a brief motivation for our definition of *spectral flow*. While our definition is, of course, logically equivalent to that of [APS] and [BW], we feel that it is technically easier to work with and easier to make rigorous. If $t \longrightarrow B_t$ for $t \in [0, 1]$ is a continuous path in $\mathcal{F}^{sa}_{*}$ then, heuristically, the spectral flow of $\{B_t\}_{t\in[0,1]}$ is the net number of eigenvalues (counted with multiplicities) which pass through 0 in the positive direction as t goes from 0 to 1. *If* we could choose a continuous family $\{E_t\}$ of finite rank projections so that each E_t were a spectral projection of B_t corresponding to some (say fixed) interval $[-a, +a]$ then the spectral flow should be the dimension of the nonnegative eigenspace of B_1E_1 (in $E_1(\mathcal{H})$) minus the dimension of the nonnegative eigenspace of B_0E_0 (in $E_0(\mathcal{H})$). We observe that the assumed continuity of $\{E_t\}$ precludes any "leakage" of eigenvalues through the boundary, $\pm a$. While, in general, it is not possible to find such a family $\{E_t\}$ on all of $[0, 1]$, one can construct them locally on subintervals: to obtain the spectral flow over $[0, 1]$ one merely adds up the flows over the subintervals. One advantage of this approach is that we do not need to restrict ourselves to any particular homotopy equivalent subsets of $\mathcal{F}^{sa}_{*}$ (see [BW] for this approach).

We use $\operatorname{sp}(\cdot)$ to denote spectrum of an element in a C^{*}-algebra.

LEMMA. *Given $B \in \mathcal{F}_*^{sa}$, there is a positive number a and a neighbourhood N of B in $\mathcal{F}_*^{sa}$ so that $S \longmapsto \chi_{[-a,a]}(S)$ is a norm-continuous, finite-rank projection-valued function on N. Where $\chi_{[-a,a]}$ denotes the characteristic function of $[-a, a]$.*

PROOF. Since B is a self-adjoint Fredholm operator, there is an $a > 0$ so that $\pm a$ are not in sp(B) and $\chi_{[-a,a]}(B)$ is a finite-rank projection. Since $\pm a$ are not in sp(B), there exists $\epsilon > 0$ so that $[-a-\epsilon, -a+\epsilon]$ and $[a-\epsilon, a+\epsilon]$ are disjoint from sp(B). The set

$$N_1 = \{S \in \mathcal{F}_*^{sa} \mid \mathrm{sp}(S) \text{ is disjoint from } [-a-\epsilon, \ -a+\epsilon] \cup [a-\epsilon, a+\epsilon]\}$$

is open and on this set the function $S \longmapsto \chi_{[-a,a]}(S)$ is norm-continuous, as $\chi_{[-a,a]}$ agrees with the function f defined below in Figure 1 on the spectrum of any S in N_1:

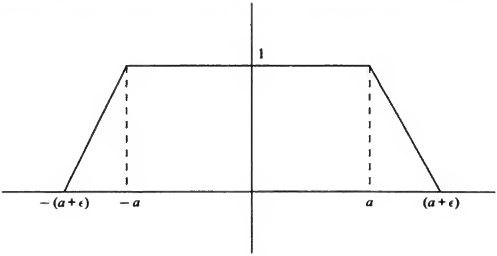

FIGURE 1: Graph of f

Let $N = \{S \in N_1 \mid \|\chi_{[-a,a]}(S) - \chi_{[-a,a]}(B)\| < 1\}$. Thus for all $S \in N$, $\dim\big(\chi_{[-a,a]}(S)\big)$ equals $\dim\big(\chi_{[-a,a]}(B)\big)$ which is finite.

NOTATION. If E is a finite-rank spectral projection for the self-adjoint operator S, let E^+ denote the projection on the subspace of $E(\mathcal{H})$ spanned by those eigenvectors for S in $E(\mathcal{H})$ having nonnegative eigenvalues.

DEFINITION. Let $B\colon [0, 1] \longrightarrow \mathcal{F}_*^{sa}$ be a continuous path. By compactness and the previous lemma, choose a partition, $0 = x_0 < x_1 < \cdots < x_n = 1$ of $[0, 1]$ and positive numbers $a_1, a_2, \ldots, a_n$ so that for each $i = 1, 2, \ldots, n$ the function $t \longmapsto E_i(t) := \chi_{[-a_i,a_i]}(B_t)$ is continuous and finite-rank on $[x_{i-1}, x_i]$. We define the spectral flow of B, sf(B) to be

$$\sum_{k=1}^{n}\Big(\dim\big(E_k^+(x_k)\big) - \dim\big(E_k^+(x_{k-1})\big)\Big).$$

PROPOSITION 2. *Spectral flow is well-defined, that is, it depends only on the continuous function $B\colon [0, 1] \longrightarrow \mathcal{F}_*^{sa}$.*

PROOF. As usual this breaks into two problems:

(1) Refining the partition using the same projection-valued functions on the refinement,

(2) Keeping the same partition, but changing the projection-valued functions.

(1) For each new point, x_*, added to the partition, the same number, $\dim\big(E_k^+(x_*)\big)$, will be both added and subtracted so that sf(B) will not change.

(2) Let $[c, d]$ be a fixed subinterval of $[0, 1]$ and

$$E_1(t) = \chi_{[-a_1,a_1]}(B_t) \quad \text{and} \quad E_2(t) = \chi_{[-a_2,a_2]}(B_t)$$

be two continuous functions on $[c, d]$ as in the definition. Without loss of generality $a_1 \geq a_2$ so that $E_1(t) \geq E_2(t)$ for all t in $[c, d]$. Moreover, a_1 and a_2 are not in sp(B_t) for any t in $[c, d]$ and so $E_1^+(t) - E_2^+(t) = \chi_{(a_2,a_1]}(B_t)$ is a *continuous* function of t on $[c, d]$, and hence of constant dimension, say n. Thus

$$\begin{aligned}\left[\dim\big(E_1^+(d)\big) - \dim\big(E_1^+(c)\big)\right] &= \left[\Big(\dim\big(E_2^+(d)\big) + n\Big) - \Big(\dim\big(E_2^+(c)\big) + n\Big)\right] \\ &= \left[\dim\big(E_2^+(d)\big) - \dim\big(E_2^+(c)\big)\right]\end{aligned}$$

and so sf(B) does not depend on the choice of $a > 0$ on the interval $[c, d]$.

REMARK. For an explicit example of a loop with spectral flow +1, see the proof of Proposition 6.

PROPOSITION 3. *Spectral flow is homotopy invariant, that is, if* $\{B_t\}$ *and* $\{B'_t\}$ *are two continuous paths with* $B_0 = B'_0$ *and* $B_1 = B'_1$ *which are homotopic via a homotopy leaving the endpoints fixed, then* sf(B) = sf(B').

PROOF. We first observe that if S and T in $\mathcal{F}_*^{sa}$ both belong to a neighbourhood N of the type given in the Lemma, then any two paths from S to T lying entirely in N have the same spectral flow. This is a trivial but crucial observation.

Now, if $H: I \times I \longrightarrow \mathcal{F}_*^{sa}$ is a homotopy from $\{B_t\}$ to $\{B'_t\}$, that is, H is continuous, $H(t, 0) = B_t$ for all t, $H(t, 1) = B'_t$ for all t, $H(0, s) = B_0 = B'_0$ for all s, and $H(1, s) = B_1 = B'_1$ for all s then, by compactness we can cover the image of H by a finite number of neighbourhoods $\{N_1, \ldots, N_k\}$ as in the Lemma. The inverse images of these neighbourhoods $\{H^{-1}(N_1), \ldots, H^{-1}(N_k)\}$ is a finite cover of $I \times I$. Thus, there exists $\epsilon_0 > 0$ (the Lebesgue number of the cover) so that any subset of $I \times I$ of diameter $\leq \epsilon_0$ is contained in some element of this finite cover if $I \times I$. Thus, if we partition $I \times I$ into a grid of squares of diameter $\leq \epsilon_0$, then the image of each square will lie entirely within some N_i. Effectively, this breaks H up into a finite sequence of "short" homotopies by restricting H to $I \times J_i$ where J_i are subintervals of I (of length $\leq \frac{\epsilon_0}{\sqrt{2}}$). These short homotopies have the added property that for fixed J_i we can choose a single partition of I so that for each subinterval J_ℓ of the partition, $H(J_\ell \times J_i)$ is contained in one of

$\{N_1, \dots, N_k\}$. By concentrating on the i-th "short homotopy" and relabelling $N_1, \dots, N_k$ if necessary we can assume H has the form indicated in Figure 2:

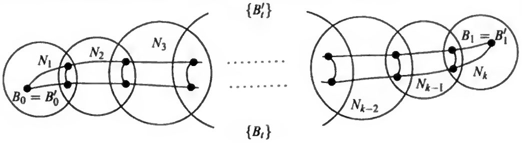

FIGURE 2.

By the observation in the first paragraph of this proof, in each of the following pairs of short paths in Figure 3, the spectral flow of the upper path equals the spectral flow of the lower path.

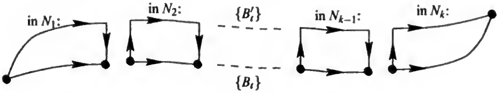

FIGURE 3.

By definition, the sum of the spectral flows of the lower paths is $\mathrm{sf}(B)$. Since the spectral flows of the vertical paths cancel in pairs, the sum of the spectral flows of the upper paths equals $\mathrm{sf}(B')$ and hence $\mathrm{sf}(B) = \mathrm{sf}(B')$.

REMARK. It is clear from our definition that spectral flow does not change under reparametrization of intervals (provided we keep the same orientation of the interval) and is additive when we combine two paths which are composable (*i.e.*, the end point of the first path is the initial point of the second path). Thus, spectral flow defines a groupoid homomorphism from the homotopy groupoid, $\mathrm{Hom}(\mathcal{F}_*^{sa})$ to $\mathbf{Z}$. By restricting to loops based at a point B_0 in $\mathcal{F}_*^{sa}$ we get a group homomorphism $\mathrm{sf}\colon \pi_1(\mathcal{F}_*^{sa}) \longrightarrow \mathbf{Z}$. Since it is easy to construct explicitly paths with spectral flow n for any $n \in \mathbf{Z}$ the map $\mathrm{sf}\colon \pi_1(\mathcal{F}_*^{sa}) \longrightarrow \mathbf{Z}$ is onto. The proof that this map is one-to-one is deeper as it depends on the homotopy equivalence $\mathcal{F}_*^{sa} \simeq U(\infty)$ of [AS] (see Step 3 below). We have almost nothing new to contribute here except a willingness to lay out the argument explicitly.

THEOREM. *Spectral flow,* $\mathrm{sf}\colon \pi_1(\mathcal{F}_*^{sa}) \longrightarrow \mathbf{Z}$ *is an isomorphism.*

PROOF. The proof breaks into three parts.

Step 1: $\mathcal{F}_*^{sa}$ is homotopy equivalent to the subspace

$$\hat{F}_*^{\infty} = \left\{B \in \mathcal{F}_*^{sa} \mid \|B\| = 1,\ \mathrm{sp}(B) \text{ is finite, and } \mathrm{sp}\big(\pi(B)\big) = \{1, -1\}\right\}.$$

Step 2: The map $B \longmapsto \exp i\pi(B+1)$ is a homotopy equivalence from $\hat{F}_*^\infty$ to $U(\infty)$.
Step 3: As mentioned above, sf: $\pi_1(\mathcal{F}_*^{sa}) \longrightarrow \mathbf{Z}$ is onto. By Step 2, $\pi_1(\mathcal{F}_*^{sa}) \cong \pi_1\big(U(\infty)\big) \cong \mathbf{Z}$, and since any onto homomorphism from $\mathbf{Z}$ to $\mathbf{Z}$ is an isomorphism, we see that sf is an isomorphism.

Our proof of Step 1 is somewhat different from [AS] or [BW]. Both of these papers use the result of [AS] that the larger subspace

$$\hat{F}_* = \big\{B \in \mathcal{F}_*^{sa} \mid \|B\| = 1,\ \mathrm{sp}\big(\pi(B)\big) = \{1,\ -1\}\big\}$$

is a deformation retract of $\mathcal{F}_*^{sa}$. We do not use this (admittedly easy) result, but instead define a mapping $\mathcal{F}_*^{sa} \longrightarrow \hat{F}_*^\infty$ which we show directly is a homotopy equivalence. In Section 17B of [BW] they make a slight misstatement: $\hat{F}_*^\infty$ cannot be a deformation retract of $\mathcal{F}_*^{sa}$: since $\hat{F}_*^\infty$ is dense in $\hat{F}_*$, any continuous map $\mathcal{F}_*^{sa} \longrightarrow \mathcal{F}_*^{sa}$ which is the identity on $\hat{F}_*^\infty$ must also be the identity on $\hat{F}_*$ and thus does *not* map into $\hat{F}_*^\infty$. We show that our mapping $\mathcal{F}_*^{sa} \longrightarrow \hat{F}_*^\infty$ is a weak deformation retraction (whose restriction to $\hat{F}_*$ is also a weak deformation retraction): this is probably what [BW] meant.

PROPOSITION 4. *There is a continuous map* $\phi\colon \mathcal{F}_*^{sa} \longrightarrow \hat{F}_*^\infty$ *which is a homotopy inverse to the injection* $i\colon \hat{F}_*^\infty \longrightarrow \mathcal{F}_*^{sa}$. *The restriction of* ϕ *to* $\hat{F}_*$ *is a homotopy inverse to the injection* $\hat{F}_*^\infty \longrightarrow \hat{F}_*$.

PROOF. We first observe that if $f\colon \mathbf{R} \longrightarrow \mathbf{R}$ is a continuous monotone function, $f(0) = 0$, and f is strictly increasing in a neighbourhood of 0 then $f\colon \mathcal{F}_*^{sa} \longrightarrow \mathcal{F}_*^{sa}$ by elementary spectral theory. Moreover, any finite convex combination of such functions is another such function. Furthermore, if f is such a function satisfying $f(t) = 1$ for all $t \geq \delta$ and $f(t) = -1$ for all $t \leq -\delta$ and $B \in \mathcal{F}_*^{sa}$ satisfies $\mathrm{sp}\big(\pi(B)\big) \cap [-\delta, \delta] = \emptyset$ then $f(B) \in \hat{F}_*^\infty$. With these observations in mind we construct our map $\phi\colon \mathcal{F}_*^{sa} \longrightarrow \hat{F}_*^\infty$.

As observed in the proof of Lemma 3, for each B in $\mathcal{F}_*^{sa}$ there is a neighbourhood N of B and a $\delta > 0$ so that for all $S \in N$, $\mathrm{sp}(S) \cap [-\delta, \delta]$ is a finite set of eigenvalues for S with finite multiplicities. Let f be the unique monotone continuous function on $\mathbf{R}$ which is +1 on $[\delta, \infty)$, -1 on $(-\infty, -\delta]$ and linear on $[-\delta, \delta]$. Then $S \longmapsto f(S)\colon N \longrightarrow \hat{F}_*^\infty$ is continuous. Since $\mathcal{F}_*^{sa}$ is a metric space, it is paracompact and so we can find a neighbourhood-finite cover $\{N_\alpha\}$ of $\mathcal{F}_*^{sa}$ and monotone continuous functions $f_\alpha\colon N_\alpha \longrightarrow \hat{F}_*^\infty$ of the type described in the first paragraph of this proof. Let $\{p_\alpha\}$ be a partition of unity subordinate to this cover. For B in $\mathcal{F}_*^{sa}$ define $\phi(B) = \Sigma_\alpha p_\alpha(B) f_\alpha(B)$. Then $\phi\colon \mathcal{F}_*^{sa} \longrightarrow \hat{F}_*^\infty$ is continuous.

One easily checks that the linear homotopy, $H(B, \lambda) = (1-\lambda)B + \lambda\phi(B)$ for $\lambda \in [0, 1]$ can be considered as a mapping: $\mathcal{F}_*^{sa} \times I \longrightarrow \mathcal{F}_*^{sa}$, or $\hat{F}_*^\infty \times I \longrightarrow \hat{F}_*^\infty$, or $\hat{F}_* \times I \longrightarrow \hat{F}_*$, and that in each case this linear homotopy does the required job.

PROPOSITION 5. *The map* $B \longmapsto \exp i\pi(B+1)$ *is a homotopy equivalence from* $\hat{F}_*^\infty$ *to* $U(\infty)$.

REMARK. This is essentially Proposition 3.3 of [AS] which is really the central difficult point of that paper and whose proof comprises Propositions 3.4, 3.5, 3.6 and 3.7 of

[AS]. The heuristic "reason" the proposition holds is that the fibre of the mapping over a point u in $U(\infty)$ is the space of nontrivial symmetries (*i.e.*, unitaries of the form $2P-1$ for P an infinite, co-infinite projection) on $\mathcal{H}_1 = \{\xi \in \mathcal{H} \mid u\xi = \xi\}$. This space is easily seen to be contractible using Kuiper's theorem and so an application of the long exact sequence of homotopy would show that the map is a weak homotopy equivalence. The argument would be finished off by applying general results on infinite dimensional manifolds (à la Palais) to conclude that the map is a homotopy equivalence. The difficult point is that the map $\hat{F}_*^\infty \longrightarrow U(\infty)$ is *not* a fibration! Nevertheless, Atiyah and Singer are able to show that the long exact sequence of homotopy does apply and so the argument can be completed. At the time of their paper they probably had the idea of quasifibrations in mind.

This completes our discussion of Step 2, and as noted above, Step 3 follows immediately so that sf is an isomorphism.

We conclude with an interesting proposition which identifies spectral flow as the composition, $\pi_1(\mathcal{F}_*^{sa}) \cong \pi_1\big(U(\infty)\big) \cong \mathbf{Z}$.

PROPOSITION 6. The following diagram is a commuting square of isomorphisms:

$$\begin{tikzcd}
\pi_1(\mathcal{F}_*^{sa}) \arrow[r, "\mathrm{sf}"] & \mathbf{Z} \\
\pi_1(\hat{F}_*^\infty) \arrow[u, "i_*"] \arrow[r] & \pi_1\big(U(\infty)\big), \arrow[u]
\end{tikzcd}$$

where as usual, the map $\pi_1\big(U(\infty)\big) \longrightarrow \mathbf{Z}$ is the winding number of the determinant.

PROOF. Since all of the maps are known to be isomorphisms and the only automorphisms of $\mathbf{Z}$ are $\pm id$, it suffices to see the diagram commutes on a single nonzero element. To this end let $1 = P_+ + P_- + P_0$ where P_+, P_- are infinite projections and P_0 is a rank 1 projection. Then $(P_+ + P_0) - P_-$ and $P_+ - (P_0 + P_-)$ are unitarily equivalent via a shift operator, u_1 which can be (explicitly) connected by a path $\{u_t\}$ to $1 = u_0$. Then $t \longrightarrow u_t[(P_+ + P_0) - P_-]u_t^*$ is a path from $(P_+ + P_0) - P_-$ to $P_+ - (P_0 + P_-)$ with spectral flow 0. On the other hand, $\{t \longrightarrow (P_+ - P_- + tP_0)\}_{t\in[-1,1]}$ is a path from $P_+ - (P_- + P_0)$ to $(P_+ + P_0) - P_-$ with spectral flow 1 and so justaposing them yields a loop based at $P_+ - (P_- + P_0)$ with spectral flow 1. The second half of this loop (with spectral flow 0) is of the form $P_t - (1 - P_t)$ and so we get:

$$\exp \pi i\Big[\big(P_t - (1 - P_t)\big) + 1\Big] = \exp 2\pi i P_t = 1,$$

the constant loop at 1. The first half of the loop yields:

$$\begin{aligned}
\exp \pi i[(P_+ - P_- + tP_0) + 1] &= \exp \pi i[2P_+ + (1 + t)P_0] \\
&= \exp\big(\pi i(1 + t)P_0\big) \\
&= P_+ + P_- + \big(\exp \pi i(1 + t)\big)P_0
\end{aligned}$$

which is a loop based at 1 with determinant $\exp \pi i(1 + t) = -\exp \pi i t$ for t in $[-1, 1]$. This clearly has winding number +1. So, we have constructed a loop in $\hat{F}_*^\infty$ with spectral flow +1 whose image in $U(\infty)$ has winding number +1, as required.

ACKNOWLEDGEMENTS. I would like to thank both the University of Victoria Operator Seminar and the Canadian Operator Symposium (1994) for being a vigorous and vocal audience. In particular, I would like to thank Ian Putnam for many stimulating conversations on this topic. I would also like to thank the referee for many useful suggestions: in particular, for the proof of Step 3 which eliminated its dependence on the final proposition.

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Canad. Math. Bull. Vol. **39** (4), 1996 pp. 468–475

THE TURAEV-VIRO INVARIANT FOR 3-MANIFOLDS IS A SUM OF THREE INVARIANTS

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ABSTRACT. We show that every Turaev-Viro invariant for 3-manifolds is a sum of three new invariants and discuss their properties. We also find a solution of a conjecture of L. H. Kauffman and S. Lins. Tables of the invariants for closed orientable 3-manifolds of complexity ≤ 3 are presented at the end of the paper.

0. **Introduction.** In 1990 Turaev and Viro obtained an infinite set of 3-manifold numerical invariants $\mathrm{TV}_{r,p}$ [7]. The invariants are parameterized by pairs (r, p) of non-negative coprime integers with $r > p > 0$ and $r > 2$.

We show that every Turaev-Viro invariant is a sum of three new invariants and discuss their properties. The paper also answers (negatively) a conjecture due to L. H. Kauffman and S. Lins [2] and studies the case $r = 4$ (for the case $r = 3$ see [7]). At the end of the paper we present tables of the invariants with $r < 8$ for closed orientable 3-manifolds of complexity ≤ 3. (For complexity theory see [3].)

The author would like to thank Prof. S. V. Matveev for his considerable amount of help in this work, A. Yu. Makovetsky and M. A. Ovchinnikov for much useful information. Special thanks to Prof. Richard B. Paine for his help in preparation of the manuscript.

1. **Definition and preliminaries.** A 2-dimensional connected polyhedron X is called *special* if it satisfies the following conditions: 1) the link of any point of X is homeomorphic to a circle, a circle with two radii, or a circle with three radii; 2) every connected component of the set of 2-manifold points of X is a 2-cell. A special polyhedron X is called a *special spine* of a compact 3-manifold M^3 with $\partial M^3 \neq \emptyset$ if there exists an imbedding $i: X \longrightarrow M^3$ such that $M^3 \searrow i(X)$, *i.e.*, M^3 collapses onto $i(X)$. A special polyhedron X is called a *special spine* of a closed 3-manifold M^3 if X is a special spine of M^3 with an open ball removed. It is known that every compact connected 3-manifold has a special spine ([1], [4]).

We shall describe elementary moves on special polyhedra. The move $\mathcal{M}$ changes a small neighborhood of some edge in a fashion indicated in Fig. 1. The move $\mathcal{M}^{-1}$ is the inverse of $\mathcal{M}$ (see [4] for details). We call two special polyhedra *$\mathcal{M}$-equivalent* if one can be obtained from the other by a finite sequence of moves $\mathcal{M}^{\pm 1}$. Let X_1 be a special spine of 3-manifold M^3 and let X_2 be a special polyhedron. Let both X_1 and X_2 have more

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Research partially supported by RFFI Grant 93-011-190

Received by the editors July 13, 1994; revised April 7, 1995.

AMS subject classification: Primary 57N10, Secondary 57M20.

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than one vertex. Then in order for X_2 also to be a special spine of M^3, it is necessary and sufficient that X_1 and X_2 be $\mathcal{M}$-equivalent (See [4], [5]).

Let X be a special spine of a 3-manifold M^3. We fix a pair (r, p) of coprime integers with $r > p > 0$ and $r > 2$. Let $\Gamma_1, \dots, \Gamma_b$ be the 2-components of X. By a *coloring* of X we mean an arbitrary mapping $\varphi\colon \{\Gamma_1, \dots, \Gamma_b\} \to \mathbf{Z}_{r-1} = \{0, 1, \dots, r-2\}$. A coloring φ is called *admissible* if for all triples of 2-components Γ_i, Γ_j, Γ_k meeting on the same edge we have:

$$2r - 4 \geq \varphi(\Gamma_i) + \varphi(\Gamma_j) + \varphi(\Gamma_k) \equiv 0 \pmod 2,$$
$$|\varphi(\Gamma_i) - \varphi(\Gamma_j)| \leq \varphi(\Gamma_k) \leq \varphi(\Gamma_i) + \varphi(\Gamma_j).$$

Let us denote the set of admissible colorings of X by $\operatorname{Adm} X$.

Let Γ_i, Γ_j, Γ_k be 2-components incident to an edge E of X and let $\varphi \in \operatorname{Adm} X$. We shall say that a triple $\big(\varphi(\Gamma_i), \varphi(\Gamma_j), \varphi(\Gamma_k)\big)$ is a *color of edge E*. There are six distinct 2-components incident to any vertex of a special spine. Suppose for a vertex v they receive under φ the values $i, j, k, l, m, n \in \mathbf{Z}_{r-1}$. A 6-tuple $\begin{pmatrix} i & j & k \\ l & m & n \end{pmatrix}$ is called a *color of vertex v* if (i, j, k) is a color of some edge incident to v and (i, l), (j, m), (k, n) are the pairs of colors of opposite 2-components incident to v.

Let us recall how to compute the Turaev-Viro invariant for compact connected 3-manifold M^3 with special spine X. For an integer $n > 0$ set

$$[n] = \sin(np\pi/r)/\sin(p\pi/r),$$
$$[n]! = [n][n-1]\cdots[2][1].$$

Set also $[0] = [0]! = 1$. For a color (i, j, k) of an edge set

$$\Delta(i,j,k) = \left(\frac{[\underline{i}+\underline{j}-\underline{k}]!\,[\underline{i}+\underline{k}-\underline{j}]!\,[\underline{j}+\underline{k}-\underline{i}]!}{[\underline{i}+\underline{j}+\underline{k}+1]!}\right)^{1/2}$$

where $\underline{i} = i/2$. Note that the expression in the round brackets presents a real number. By the square root $x^{1/2}$ of a real number x we mean the positive root of $|x|$ multiplied by $\sqrt{-1}$ if $x < 0$.

Let $\begin{pmatrix} i & j & k \\ l & m & n \end{pmatrix}$ be a color of some vertex v. A *symbol* of v is computed by the following formula

$$\begin{aligned} |T_v^\varphi| &= \begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix} \\ &= (\sqrt{-1})^{-(i+j+k+l+m+n)} \Delta(i,j,k)\Delta(i,m,n)\Delta(j,l,n)\Delta(k,l,m) \begin{bmatrix} i & j & k \\ l & m & n \end{bmatrix}, \end{aligned}$$

where

$$\begin{aligned} \begin{bmatrix} i & j & k \\ l & m & n \end{bmatrix} = \sum_z (-1)^z [z+1]!\{&[z-\underline{i}-\underline{j}-\underline{k}]!\,[z-\underline{i}-\underline{m}-\underline{n}]!\,[z-\underline{j}-\underline{l}-\underline{n}]!\,[z-\underline{k}-\underline{l}-\underline{m}]! \\ &\times [\underline{i}+\underline{j}+\underline{l}+\underline{m}-z]!\,[\underline{i}+\underline{k}+\underline{l}+\underline{n}-z]!\,[\underline{j}+\underline{k}+\underline{m}+\underline{n}-z]!\}^{-1}. \end{aligned}$$

Here z runs through the non-negative integers such a way that all expressions in the square brackets are non-negative. For $i \in \mathbf{Z}_{r-1}$ put

$$w_i(r,p) = (\sqrt{-1})^i [i+1]^{1/2}.$$

For $\varphi \in \operatorname{Adm} X$ put

$$|X|_\varphi(r,p) = \prod_{i=1}^{b} w^2_{\varphi(\Gamma_i)} \prod_{j=1}^{d} |T^\varphi_{v_j}|,$$

where $v_1, \dots, v_d$ are the vertices of X. The number

$$\mathrm{TV}_{r,p}(M^3) = w^{-2\chi(X)} \sum_{\varphi \in \operatorname{Adm} X} |X|_\varphi(r,p)$$

is the *Turaev-Viro invariant* for 3-manifold M^3, where $w = \pm\sqrt{2r}/\big(2\sin(p\pi/r)\big)$ and $\chi(X)$ is Euler characteristic of X. (See [7] for details.)

The set of 2-components that receive odd colors under an admissible coloring φ forms a closed surface embedded in X. We shall denote this surface by $S(\varphi)$. Present the set $\operatorname{Adm} X$ as a disjoint union of subsets $\operatorname{Adm}_0 X$, $\operatorname{Adm}_1 X$, and $\operatorname{Adm}_2 X$, where

0) $\varphi \in \operatorname{Adm}_0 X \Leftrightarrow (\varphi \in \operatorname{Adm} X) \,\&\, \big(S(\varphi) = \emptyset\big)$;

1) $\varphi \in \operatorname{Adm}_1 X \Leftrightarrow (\varphi \in \operatorname{Adm} X) \,\&\, \Big(\chi\big(S(\varphi)\big) \equiv 1 \pmod 2\Big)$;

2) $\varphi \in \operatorname{Adm}_2 X \Leftrightarrow (\varphi \in \operatorname{Adm} X) \,\&\, \big(S(\varphi) \neq \emptyset\big) \,\&\, \Big(\chi\big(S(\varphi)\big) \equiv 0 \pmod 2\Big)$.

Here $\chi\big(S(\varphi)\big)$ is Euler characteristic of surface $S(\varphi)$. If $\varphi \in \operatorname{Adm}_N X$, where $N = 0$, 1, or 2, then we say that N is the *index of coloring* φ.

2. **Main theorem.**

THEOREM 2.1. *Let M^3 be a compact 3-manifold and X_1 be its special spine. Then the numbers*

$$|M^3|_0(r,p) = \sum_{\varphi \in \operatorname{Adm}_0 X_1} |X_1|_\varphi(r,p),$$
$$|M^3|_1(r,p) = \sum_{\varphi \in \operatorname{Adm}_1 X_1} |X_1|_\varphi(r,p),$$
$$|M^3|_2(r,p) = \sum_{\varphi \in \operatorname{Adm}_2 X_1} |X_1|_\varphi(r,p)$$

are invariants of M^3 and

$$\mathrm{TV}_{r,p}(M^3) = w^{-2\chi(X)}\big(|M^3|_0(r,p) + |M^3|_1(r,p) + |M^3|_2(r,p)\big).$$

PROOF. Let X_2 denote the spine, which is obtained from X_1 by a single elementary move $\mathcal{M}$. It suffices to prove that

$$\sum_{\varphi \in \operatorname{Adm}_N X_1} |X_1|_\varphi(r,p) = \sum_{\varphi \in \operatorname{Adm}_N X_2} |X_2|_\varphi(r,p)$$

for every $N \in \{0, 1, 2\}$.

By definition of the $\mathcal{M}$-move, there exist imbeddings $i_1 : P_1 \longrightarrow X_1$ and $i_2 : P_2 \longrightarrow X_2$ such that $X_1 - i_1(P_1) = X_2 - i_2(P_2)$, where P_1 and P_2 are the polyhedra in Fig. 1. It is evident that for any $\varphi_1 \in \operatorname{Adm} X_1$ and $\varphi_2 \in \operatorname{Adm} X_2$ such that $\varphi_1|_{X_1 - i_1(P_1)} = \varphi_2|_{X_2 - i_2(P_2)}$ the indices of colorings φ_1 and φ_2 are the same, *i.e.*, there exists a unique number $N \in \{0, 1, 2\}$ such that $\varphi_1 \in \operatorname{Adm}_N X_1$ and $\varphi_2 \in \operatorname{Adm}_N X_2$. The remaining part of the Proof coincides with the Proof of 4.4.B from [7]. ■

The following Lemma shows that the invariants $|M^3|_N(r, p)$ are not independent.

LEMMA 2.2. *Let M^3 be a compact orientable 3-manifold. Then*

1) *For all parameters (r, p) we have $|M^3|_N(r, p) = (-1)^N |M^3|_N(r, r-p)$, where $N \in \{0, 1, 2\}$.*
2) $|M^3|_0(3, 1) = |M^3|_0(3, 2) = 1$.

PROOF. 1) It is easily proved that for any orientable 3-manifold M^3 with special spine X and for any $\varphi \in \operatorname{Adm} X$ we have

$$|X|_\varphi(r,p) = (-1)^{\chi(S(\varphi))} |X|_\varphi(r, r-p) \tag{1}$$

(for proof see [6]). It remains to check that $\chi\big(S(\varphi)\big) \equiv N \pmod 2$.

2) If $r = 3$ then $\operatorname{Adm}_0 X = \{\varphi_0\}$, where φ_0 is the zero–coloring. It is known that $|X|_{\varphi_0}(3, 1) = 1$. ■

3. **The Kauffman-Lins conjecture. The case r=4.** Throughout this section, $\mathrm{TV}^*_{r,p}$ will be the Turaev-Viro invariant without the factor $w^{-2\chi(X)}$, *i.e.*,

$$\mathrm{TV}^*_{r,p}(M^3) = \sum_{\varphi \in \operatorname{Adm} X} |X|_\varphi(r,p).$$

(Without this factor the invariant depends on removing open balls from M^3. But this fact is irrelevant here.)

In [2] L. H. Kauffman and S. Lins put forward the following conjecture.

CONJECTURE. Consider an arbitrary closed 3-manifold M^3, and let X be a special spine for M^3. Let n_e be the number of closed surfaces contained in X that have even Euler characteristic and n_o the number of closed surfaces in X that have odd Euler characteristic. Then (i) either $n_e = n_o$, or $n_o = 0$, and this alternative is invariant for every special spine of M^3; (ii) $n_o = 0 \Leftrightarrow \mathrm{TV}^*_{4,1}(M^3) = \mathrm{TV}^*_{4,3}(M^3) \in \mathbf{Z}$.

Note that if M^3 is an orientable manifold, then part (i) of the conjecture follows from 8.3–8.4 in [7]. If M^3 is a nonorientable manifold, then there exists a counter example offered by S. V. Matveev. It is the manifold $\mathbf{R}P^2 \times S^1$; a neighborhood of the 1-skeleton of a special spine for $\mathbf{R}P^2 \times S^1$ is shown on Fig. 2a. For this spine we have: $n_o = 1$ and $n_e = 3$. This disproves the part (i) of the conjecture. (Let us give some explanation of Fig. 2a. The manifold $\mathbf{R}P^2 \times S^1$ with open ball B^3 removed collapses onto $M \times S^1 \cup_{\partial M} D^2$, where M is Möbius band and D^2 is a disk attached along ∂M. The manifold $M \times S^1$ collapses onto torus $S^1 \times S^1$, so $\mathbf{R}P^2 \times S^1 - B^3$ collapses onto $S^1 \times S^1$ with disk D^2,

	$L_{3.1}$	S^3	$L_{5.2}$	$L_{4.1}$	$\mathbf{R}P^3$	$L_{5.1}$	$L_{7.2}$	$L_{8.3}$	S^3/Q_8	$L_{6.1}$	$L_{9.2}$	$L_{10.3}$	$L_{11.4}$	$L_{12.5}$	$L_{13.5}$	S^3/Q_{12}
(3,1)	0.500	0.500	0.500	1.000	0.000	0.500	0.500	1.000	2.000	0.000	0.500	0.000	0.500	1.000	0.500	1.000
(3,2)	0.500	0.500	0.500	1.000	1.000	0.500	0.500	1.000	2.000	1.000	0.500	1.000	0.500	1.000	0.500	1.000
(4,1)	0.250	0.250	0.250	0.500	0.146	0.250	0.250	1.000	2.500	0.853	0.250	0.853	0.250	0.500	0.250	0.500
(4,3)	0.250	0.250	0.250	0.500	0.853	0.250	0.250	1.000	2.500	0.146	0.250	0.146	0.250	0.500	0.250	0.500
(5,1)	0.362	0.138	0.000	0.276	0.000	0.500	0.362	0.724	2.553	0.000	0.138	0.000	0.138	0.724	0.362	1.276
(5,4)	0.362	0.138	0.000	0.276	0.724	0.500	0.362	0.724	2.553	0.276	0.138	0.000	0.138	0.724	0.362	1.276
(5,2)	1.382	0.362	0.000	0.724	0.276	0.500	0.138	0.276	3.447	0.724	0.362	0.000	0.362	0.276	0.138	1.724
(5,3)	1.382	0.362	0.000	0.724	0.000	0.500	0.138	0.276	3.447	0.000	0.362	0.000	0.362	0.276	0.138	1.724
(6,1)	0.250	0.083	0.083	0.333	0.045	0.083	0.083	0.333	2.333	0.500	0.250	0.622	0.083	1.000	0.083	1.333
(6,5)	0.250	0.083	0.083	0.333	0.622	0.083	0.083	0.333	2.333	0.500	0.250	0.045	0.083	1.000	0.083	1.333
(7,1)	0.175	0.054	0.272	0.349	0.000	0.272	0.000	0.108	2.043	0.000	0.272	0.000	0.175	0.543	0.054	0.758
(7,6)	0.175	0.054	0.272	0.349	0.543	0.272	0.000	0.108	2.043	0.108	0.272	0.349	0.175	0.543	0.054	0.758
(7,2)	0.272	0.175	0.054	0.543	0.108	0.054	0.000	0.349	4.268	0.349	0.054	0.543	0.272	0.108	0.175	0.806
(7,5)	0.272	0.175	0.054	0.543	0.000	0.054	0.000	0.349	4.268	0.000	0.054	0.000	0.272	0.108	0.175	0.806
(7,3)	0.054	0.272	0.175	0.108	0.000	0.175	0.000	0.543	3.689	0.000	0.175	0.000	0.054	0.349	0.272	1.436
(7,4)	0.054	0.272	0.175	0.108	0.349	0.175	0.000	0.543	3.689	0.543	0.175	0.108	0.054	0.349	0.272	1.436

TABLE 1

attached to it such that ∂D^2 winds twice around $S^1 \times \{*\}$. There is an isotopy of ∂D^2 in the torus $S^1 \times S^1$ transforming our polyhedron to special one with neighborhood of the 1-skeleton pictured on Fig. 2a. Union of 2-components marked by I, II, III is the torus $S^1 \times S^1$, unmarked curve is the boundary of the disk D^2.)

Let us consider the manifold S^3/Q_{16} (A neighborhood of the 1-skelton of a special spine for S^3/Q_{16} is shown on Fig. 2b, see [3]). We can see that $\mathrm{TV}^*_{4.1}(S^3/Q_{16}) = \mathrm{TV}^*_{4.3}(S^3/Q_{16}) = 6$, but $n_o \neq 0$ for this manifold. So the implication $\Leftarrow$ of part (ii) in the conjecture is not true. But if $n_o = 0$ then $\mathrm{TV}^*_{4.1}(M^3) = \mathrm{TV}^*_{4.3}(M^3) \in \mathbf{Z}$. It follows from the next Lemma. (Let us remark that $\mathrm{TV}^*_{r.p}(M^3) = \mathrm{TV}^*_{r.r-p}(M^3)$ in the case $n_o = 0$ for any (r, p) and any compact orientable 3-manifold M^3. It follows from the equation (1)).

Let us denote by $S_i(\varphi)$ the union of 2-components $\Gamma_j \subset X$ such that $\varphi(\Gamma_j) = i$.

LEMMA 3.1. *Let X be a special spine of a compact 3-manifold M^3, let $r = 4$, and $\varphi \in \mathrm{Adm}X$. Then*

$$|X|_\varphi(4,1) = (-\sqrt{2})^{\chi(S_1(\varphi))}(-1)^n, \tag{2}$$

$$|X|_\varphi(4,3) = (\sqrt{2})^{\chi(S_1(\varphi))}(-1)^n, \tag{2'}$$

where n is the number of vertices in the graph $G = \partial S_2(\varphi)$.

PROOF. Values of symbols for vertices and weights for 2-cells of the spine X in the case $r = 4$ are given in [2]. Their products give necessary equalities. ■

COROLLARY 3.2. *If M^3 is a compact orientable 3-manifold, then*

$$|M^3|_0(4,1) = |M^3|_0(4,3) = \mathrm{TV}^*_{3.2}(M^3).$$

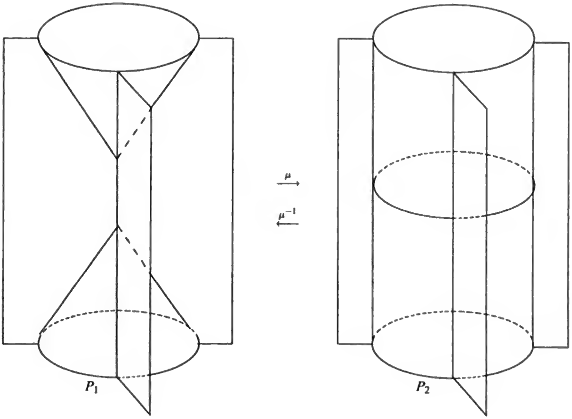
	$L_{3,1}$	S^3	$L_{5,2}$	$L_{4,1}$	$\mathbf{R}P^3$	$L_{5,1}$	$L_{7,2}$	$L_{8,3}$	S^3/Q_8	$L_{6,1}$	$L_{9,2}$	$L_{10,3}$	$L_{11,4}$	$L_{12,5}$	$L_{13,5}$	S^3/Q_{12}
(3,1)0	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
(3,1)1	0.000	0.000	0.000	0.000	−1.000	0.000	0.000	0.000	0.000	−1.000	0.000	−1.000	0.000	0.000	0.000	0.000
(3,1)2	0.000	0.000	0.000	1.000	0.000	0.000	0.000	1.000	3.000	0.000	0.000	0.000	0.000	1.000	0.000	1.000
(4,1)0	1.000	1.000	1.000	2.000	2.000	1.000	1.000	2.000	4.000	2.000	1.000	2.000	1.000	2.000	1.000	2.000
(4,1)1	0.000	0.000	0.000	0.000	−1.414	0.000	0.000	0.000	0.000	1.414	0.000	1.414	0.000	0.000	0.000	0.000
(4,1)2	0.000	0.000	0.000	0.000	0.000	0.000	0.000	2.000	6.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
(5,1)0	2.618	1.000	0.000	1.000	2.618	3.618	2.618	2.618	4.618	1.000	1.000	0.000	1.000	2.618	2.618	4.618
(5,1)1	0.000	0.000	0.000	0.000	−2.618	0.000	0.000	0.000	0.000	−1.000	0.000	0.000	0.000	0.000	0.000	0.000
(5,1)2	0.000	0.000	0.000	1.000	0.000	0.000	0.000	2.618	13.854	0.000	0.000	0.000	0.000	2.618	0.000	4.618
(5,2)0	0.382	1.000	0.000	1.000	0.382	1.382	0.382	0.382	2.382	1.000	1.000	0.000	1.000	0.382	0.382	2.382
(5,2)1	0.000	0.000	0.000	0.000	0.382	0.000	0.000	0.000	0.000	1.000	0.000	0.000	0.000	0.000	0.000	0.000
(5,2)2	0.000	0.000	0.000	1.000	0.000	0.000	0.000	0.382	7.146	0.000	0.000	0.000	0.000	0.382	0.000	2.382
(6,1)0	3.000	1.000	1.000	4.000	4.000	1.000	1.000	4.000	10.000	6.000	3.000	4.000	1.000	6.000	1.000	10.000
(6,1)1	0.000	0.000	0.000	0.000	−3.464	0.000	0.000	0.000	0.000	0.000	0.000	3.464	0.000	0.000	0.000	0.000
(6,1)2	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	18.000	0.000	0.000	0.000	0.000	6.000	0.000	6.000
(7,1)0	3.247	1.000	5.049	3.247	5.049	5.049	0.000	1.000	9.494	1.000	5.049	3.247	3.247	5.049	1.000	7.049
(7,1)1	0.000	0.000	0.000	0.000	−5.049	0.000	0.000	0.000	0.000	−1.000	0.000	−3.247	0.000	0.000	0.000	0.000
(7,1)2	0.000	0.000	0.000	3.247	0.000	0.000	0.000	1.000	28.482	0.000	0.000	0.000	0.000	5.049	0.000	7.049
(7,2)0	1.555	1.000	0.308	1.555	0.308	0.308	0.000	1.000	6.110	1.000	0.308	1.555	1.555	0.308	1.000	2.308
(7,2)1	0.000	0.000	0.000	0.000	0.308	0.000	0.000	0.000	0.000	1.000	0.000	1.555	0.000	0.000	0.000	0.000
(7,2)2	0.000	0.000	0.000	1.555	0.000	0.000	0.000	1.000	18.330	0.000	0.000	0.000	0.000	0.308	0.000	2.308
(7,3)0	1.198	1.000	0.643	0.198	0.643	0.643	0.000	1.000	3.396	1.000	0.643	0.198	0.198	0.643	1.000	2.643
(7,3)1	0.000	0.000	0.000	0.000	−0.643	0.000	0.000	0.000	0.000	−1.000	0.000	−0.198	0.000	0.000	0.000	0.000
(7,3)2	0.000	0.000	0.000	0.198	0.000	0.000	0.000	1.000	10.188	0.000	0.000	0.000	0.000	0.643	0.000	2.643

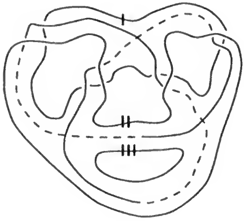
TABLE 2

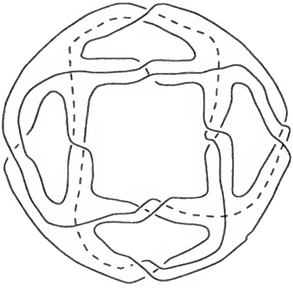
PROOF. Let X be a special spine of M^3. If $\varphi \in \mathrm{Adm}_0 X$, then $\chi\big(S_1(\varphi)\big)$ and n in (2) and (2′) are equal to 0, so we have $|M^3|_0(4, 1) = |M^3|_0(4, 3) = |\,\mathrm{Adm}_0 X|$. Consider the coloring $\underline{\varphi}\colon \{\Gamma_1, \dots, \Gamma_b\} \longrightarrow \mathbf{Z}_2$ such that $\underline{\varphi}(\Gamma_i) = \varphi(\Gamma_i)/2$ for every $1 \leq i \leq b$. The mapping $\varphi \longmapsto \underline{\varphi}$ gives a bijection between $\mathrm{Adm}_0 X$ in the case $r = 4$ and $\mathrm{Adm}\, X$ in the case $r = 3$. The equation $\mathrm{TV}'_{3,2}(M^3) = |\,\mathrm{Adm}\, X|$ follows from 8.3 in [7]. ■

4. **The tables.** We computed the values for invariants $\mathrm{TV}_{r,p}(M^3)$ and $|M^3|_N(r, p)$ with $r \leq 7$ for all prime closed orientable 3-manifolds of complexity ≤ 6. The computed values allowed us to conjecture that the sums $\sum_{p<r/2} |M^3|_0(r, p)$ and $\sum_{p<r/2} |M^3|_2(r, p)$ for any closed orientable 3-manifold M^3 are integers. Analysis of computation shows that the invariants with $r \leq 7$ identify 45 manifolds from 61 of complexity < 6. There are two pairs and four triples of 3-manifolds with the same computed invariants.

We present the tables of invariants for the prime closed orientable 3-manifolds of complexity < 4. The first table contains the original Turaev-Viro invariants $\mathrm{TV}_{r,p}(M^3)$.




a) $\mathbb{R}P^2 \times S^1$


b) S^3/Q^{16}

The second table contains the invariants $|M^3|_N(r,p)$ (see section 2). The names we have chosen for the manifolds are taken from [3]. The numbers in the brackets in the first columns of tables are the parameters (r,p); the number 0, 1, or 2 after the brackets in the first column of the second table shows what kind of three invariants $|M^3|_0(r,p)$, $|M^3|_1(r,p)$, or $|M^3|_2(r,p)$ is presented in this line. The second table does not contain invariants with parameters $p \geq r/2$ because it has been proved in Lemma 2.2 that $|M^3|_N(r,p) = (-1)^N|M^3|_N(r,r-p)$.

The invariants from the first table and the second one are related by the equality

$$\mathrm{TV}_{r,p}(M^3) = \left(2\sin^2(p\pi/r)/r\right) \times \left(|M^3|_0(r,p) + |M^3|_1(r,p) + |M^3|_2(r,p)\right).$$

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Canad. Math. Bull. Vol. 39 (4), 1996 pp. 476–485

WHEN X^* IS A P'-SPACE

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ABSTRACT. In [7,3.1] the authors show that if a space X is realcompact and locally compact, then X^* is a P'-space. In this paper we show that the hypothesis of realcompactness can be weakened. We also look at other conditions on X that are sufficient to guarantee that X^* is a P'-space.

0. **Introduction.** A space X is a P'-space if every zero-set of X has non-empty interior in X or, equivalently, if every zero-set is regular closed in X. In [7,3.1] Fine and Gillman show that if a space X is locally compact and realcompact, then $\beta X - X$ is a P'-space. We show that the hypothesis that X is realcompact can be reduced. We introduce the class of p-realcompact spaces, a class that lies strictly between the c-realcompact spaces and the nearly realcompact spaces. We then show that locally compact p-realcompact spaces have growths that are P'-spaces.

Along the way to producing this result, we introduce some local conditions on a zero-set of a locally compact nearly realcompact space X that are necessary and sufficient to guarantee that the zero-set of its extension will trace to a set with empty interior in X^*.

1. **Definitions and Preliminaries.** There are a number of lines along which realcompactness has been generalized in the literature. By a "line" we mean a sequence of properties of a space, each weaker than its predecessor. One of these lines is an "isocompactness" line that characterizes spaces by which of their closed subsets are compact. The line we are mainly interested in concerns the structure of $\beta X - X$. The properties in this line generalize the characterization of realcompactness described in Proposition 1.0 below. It is in this line that we find the property that guarantees (in the presence of local compactness) that $\beta X - X$ is a P'-space. We will also be concerned with the "isocompactness line" and will discuss where the two lines intersect. Thus we need the following definitions and preliminary results.

Since we are concerned with the structure of the growth of a space X, that is, with $X^*(= \beta X - X)$, we will naturally assume that all our spaces are Tychonoff. As usual, we denote the collection of all real-valued continuous (resp. bounded continuous) functions on a space X by $C(X)$ (resp. $C^*(X)$). A zero-set Z of X is a set of the form $Z(f) = f^{\leftarrow}\{0\}$ where $f \in C(X)$. A cozero-set is the complement of a zero-set. A *z-ultrafilter* on X is a

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Received by the editors June 26, 1995.

AMS subject classification: Primary: 54D60; Secondary: 54D35, 54D45.

Key words and phrases: P'-space, c-realcompact, p-realcompact, nearly realcompact, locally compact, hyperisocompact, strongly isocompact, (strongly) relatively pseudocompact, well separated.

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maximal filter in the collection of zero-sets of X. We denote the natural numbers by $\mathbb{N}$ and the non-negative integers by ω.

We assume that the reader is familiar with the basic theory of both the Stone-Čech compactification βX and the Hewitt realcompactification υX of a Tychonoff space X as described in [8]. In particular we think of points of βX as z-ultrafilters on X and points of υX as z-ultrafilters on X with the countable intersection property (and thus a space X is realcompact if and only if every z-ultrafilter with the countable intersection property converges). We will often use the following well known characterization of points in $\beta X - \upsilon X$.

PROPOSITION 1.0. *A point p in βX is in $\beta X - \upsilon X$ if and only if there is $g \in C(\beta X)$ with $g(p) = 0$ and $g > 0$ on X.*

Subsets A and B of X are *completely separated* in X if there is $f \in C(X)$ with $f^{\to}(A) \subset \{0\}$ and $f^{\to}(B) \subset \{1\}$. Completely separated subsets of X have disjoint closures in βX. A subset $A \subset X$ is *well embedded* in X if A is completely separated in X from every disjoint zero-set of X. It is well known that zero-sets of X are well embedded in X. A set $A \subset X$ is *relatively pseudocompact* (resp. *strongly relatively pseudocompact*) in X if every function in $C(X)$ (resp. for every cozero-set neighborhood P of A, every function in $C(P)$) is bounded on A. A space X is *isocompact* (resp. *strongly isocompact*) (resp. *hyperisocompact*) if every countably compact (resp. every strongly relatively pseudocompact) (resp. every relatively pseudocompact) closed subset of X is compact. Clearly hyperisocompact implies strongly isocompact implies isocompact.

Unfortunately there is some confusion in the literature about these terms. The term "strongly isocompact" was used in [1] to mean "hyperisocompact". The terminology used here follows [3].

2. p-realcompact spaces.

We begin with the definition of some of the properties in the "structure of $\beta X - X$" line.

A space X is *c-realcompact* (see [9]) if for every $p \in X^*$, there is a decreasing sequence $\langle A_n : n \in \omega \rangle$ of regular closed subsets of βX with $p \in \bigcap_{n\in\omega} A_n$ while $\bigcap_{n\in\omega}(A_n \cap X) = \emptyset$. X is *nearly realcompact* (see [2]) if $\beta X - \upsilon X$ is dense in $\beta X - X$. Finally we define X to be *p-realcompact* if every zero-set of βX that meets X^* meets $\beta X - \upsilon X$.

We have the following implications already available. The first implication is an easy consequence of Proposition 1.0 and the second is due to Blair and van Douwen and is proved in [11,14.3].

PROPOSITION 2.0. *Realcompact spaces are c-realcompact, and c-realcompact spaces are nearly realcompact.*

We show next where p-realcompactness fits into this scheme, but first we need the following facts from elsewhere.

PROPOSITION 2.1 [2,1.4]. *If G is open in X, then the following are equivalent.*

(1) G is relatively pseudocompact in X.

(2) If $\langle F_n : n \in \omega\rangle$ is a decreasing sequence of regular closed subsets of X with $F_n \cap G \neq \emptyset$ for all $n \in \omega$, then $\bigcap_{n\in\omega} F_n \neq \emptyset$.

We use the following two propositions often. The proof of the first is trivial and the proof of the second can be found in [3].

PROPOSITION 2.2. *For all $f \in C(X)$, if $Z(f^\beta) \subset \upsilon X$, then $\mathrm{cl}_{\beta X} Z(f) = Z(f^\beta)$.*

PROPOSITION 2.3 [3,2.6]. *If A is a subspace of the space X, then the following are equivalent.*

(1) A is relatively pseudocompact in X.

(2) $\mathrm{cl}_{\beta X} A \subset \upsilon X$.

PROPOSITION 2.4. *c-realcompact spaces are p-realcompact and p-realcompact spaces are nearly realcompact.*

PROOF. The second implication is trivial. We prove the first. Let X be a c-realcompact space and let $f\colon \beta X \longrightarrow [0,1]$ with $p \in Z(f) \cap X^*$. There is a decreasing sequence $\langle A_n : n \in \omega\rangle$ of regular closed sets in βX with $p \in \bigcap_{n\in\omega} A_n$ and $\bigcap_{n\in\omega}(A_n \cap X) = \emptyset$. Assume that $Z(f) \subset \upsilon X$. Then by Proposition 2.2, $\mathrm{cl}_{\beta X}\big(Z(f) \cap X\big) = Z(f)$.

For all $n \in \mathbb{N}$, let

$$B_n = (\mathrm{int}_{\beta X} A_n) \cap f^{\leftarrow}(0, 1/n) \cap X.$$

We will show that $B_n \neq \emptyset$ for all $n \in \mathbb{N}$. Suppose, on the contrary, that there is $N \in \mathbb{N}$ with $B_N = \emptyset$. Now for all $n \geq N$, $f^{\leftarrow}[0, 1/n) \cap \mathrm{int}_X(A_n \cap X) \neq \emptyset$, and so by our assumption, $\emptyset \neq \mathrm{int}_X(A_n \cap X) \cap f^{\leftarrow}[0, 1/n) \subset Z(f)$ for all $n \geq N$. Thus we have, for all $n \geq N$, $A_n \cap \mathrm{int}_X\big(Z(f) \cap X\big) \neq \emptyset$. Now by Proposition 2.3, $\mathrm{int}_X\big(Z(f) \cap X\big)$ is relatively pseudocompact in X and so by Proposition 2.1, $\emptyset \neq \bigcap_{n\geq N}(A_n \cap X) \subset \bigcap_{n\in\mathbb{N}}(A_n \cap X)$, a contradiction. We conclude that $B_n \neq \emptyset$ for all $n \in \mathbb{N}$.

We define, recursively, sequences $\langle Y_n : n \in \mathbb{N}\rangle$ of zero-sets of X, $\langle P_n : n \in \mathbb{N}\rangle$ of cozero-sets of X, $\langle q_n : n \in \mathbb{N}\rangle \subset X$, $\langle r_n : n \in \mathbb{N}\rangle$ and $\langle s_n : n \in \mathbb{N}\rangle \subset \mathbb{R}$, satisfying:

(1) $0 < s_{n+1} < r_n < s_n < \frac{1}{n+1}$ for all $n \in \mathbb{N}$; and

(2) $q_n \in Y_n \subset P_n \subset f^{\leftarrow}(r_n, s_n) \cap \mathrm{int}_X A_n$ for all $n \in \mathbb{N}$.

Let $s_0 = \frac{1}{2}$. Assume now that $n \in \mathbb{N}$ and that for all $j \leq n$ the sequences have been constructed and that s_n has been defined. There is $k \in \mathbb{N}$ with $\frac{1}{k} < \min\{s_n, \frac{1}{n+1}\}$. Pick $q_n \in B_k$. Then $0 < f(q_n) < \min\{s_n, \frac{1}{n+1}\}$. Pick r_n with $0 < r_n < f(q_n)$. Finally pick a zero-set neighborhood Y_n and a cozero-set neighborhood P_n of q_n such that

$$Y_n \subset P_n \subset f^{\leftarrow}(r_n, s_n) \cap \mathrm{int}_X A_n.$$

Now pick $s_{n+1} < \min\{r_n, \frac{1}{n+2}\}$ with $0 < s_{n+1}$. This completes the recursion.

Let $Z_n = \bigcup_{j>n} Y_j$ for all $n \in \mathbb{N}$ and note that Z_n is a zero-set of X. Let q be a z-ultrafilter on X containing $\{Z_n : n \in \mathbb{N}\}$. Since $\bigcap_{n\in\mathbb{N}} Z_n \subset \bigcap_{n\in\omega}(A_n \cap X) = \emptyset$, $q \in \beta X - \upsilon X$. Clearly $f(q) = 0$. This contradiction completes the proof.

We denote the extension of the function $f \in C^*(X)$ to βX by f^β.

PROPOSITION 2.5 [3,2.7]. *If A is a subspace of X, then the following are equivalent.*

(1) A is strongly relatively pseudocompact in X.

(2) A is relatively pseudocompact and well embedded in X.

Obviously every relatively pseudocompact zero-set is strongly relatively pseudocompact. We denote the extension of the function $f \in C^*(X)$ to βX by f^β.

PROPOSITION 2.6. *If every relatively pseudocompact zero-set of X is compact, then X is p-realcompact. Hence strongly isocompact spaces are p-realcompact.*

PROOF. Let $Z(f^\beta)$ be a zero-set of βX and suppose that $Z(f^\beta) \subset \upsilon X$. By Proposition 2.2, $\mathrm{cl}_{\beta X} Z(f) = Z(f^\beta)$ and so by Proposition 2.3 $Z(f)$ is relatively pseudocompact and hence compact. Therefore $Z(f^\beta) \cap X^* = \emptyset$.

COROLLARY 2.7. *Topologically complete spaces are p-realcompact.*

PROOF. Topologically complete spaces are hyperisocompact (see [5,3.1]).

EXAMPLES 2.8. The following examples show that p-realcompact spaces need not be c-realcompact and that nearly realcompact spaces need not be p-realcompact.

(1) A nearly realcompact space that is not p-realcompact. Let $X = \mathbf{Q} \times \omega_1$ where $\mathbf{Q}$ is the set of rationals. By [2,1.11], X is nearly realcompact. X is not p-realcompact since the set $\{0\} \times \omega_1$ ($= Z(f)$ where $f(\langle x, \alpha \rangle) = |x|$) is a non-compact zero-set of X but $Z(f^\beta) = \{0\} \times [0, \omega_1] \subset \upsilon X$.

(2) A p-realcompact space that is neither c-realcompact nor isocompact.

Let T be the Tychonoff Plank $([0,\omega_1] \times [0,\omega]) - \{\langle \omega_1, \omega \rangle\}$ and let $A = \{x_\alpha^\beta : \alpha, \beta \in \omega_1\}$. Let $Y = T \cup A$. We topologize Y by isolating all points of A, giving all points of T that do not lie on the top edge of T their usual neighborhoods in T and letting points of the top edge of T have their usual neighborhoods in T together with sequences from A that converge to each point of the neighborhood. That is, a basic neighborhood of $\langle \alpha, \omega \rangle$ will be of the form $U \cup \{x_\delta^\gamma : \gamma > \beta \text{ and } \langle \delta, \omega \rangle \in U\}$ where U is a usual neighborhood in T and $\beta \in \omega_1$.

Note that if Z is a non-compact zero-set of Y, then $|Z \cap A| \geq \omega$. Every countable subset of A is clopen and so no infinite subset of A is relatively pseudocompact in Y. Thus if Z is a non-compact zero-set of Y then Z is not relatively pseudocompact. We conclude, by Proposition 2.6, that Y is p-realcompact.

To see that Y is not c-realcompact, we note first that T is C-embedded in Y and so $p = \langle \omega_1, \omega \rangle \in \upsilon T \subset \upsilon Y$. But if $p \in \bigcap_{n \in \omega} A_n$ where each A_n is regular closed in βY, then $\bigcap_{n \in \omega} (A_n \cap Y) \neq \emptyset$.

Finally, the top edge of T witnesses that Y is not isocompact.

(3) A c-realcompact space X that is not isocompact.

Let X be the Tychonoff plank T with a copy of $(0,1) = G_n$ inserted between $\langle \omega_1, n \rangle$ and $\langle \omega_1, n+1 \rangle$ for each $n \in \omega$. That is, we identify the 0 of the n-th copy of $[0,1]$ with $\langle \omega_1, n \rangle$ and the 1 of $[0,1]$ with $\langle \omega_1, n+1 \rangle$. Every point of X^* is in $\bigcap_{n \in \omega} \mathrm{cl}_{\beta X}(\bigcup_{j > n} G_j)$ and so X is c-realcompact. Clearly X is not isocompact. (This example appears in [2,1.15].)

We remark that p-realcompactness does not have nice hereditary properties. In Example (3) T is a regular closed C-embedded zero-set of a p-realcompact space that is not p-realcompact.

We give next a couple of sufficient conditions for a space to be p-realcompact. We use the notation

$$\mathrm{Ex}(U) = \beta X - \mathrm{cl}_{\beta x}(X - U)$$

for open subsets U of X.

PROPOSITION 2.9. *If every point in $\upsilon X - X$ has neighborhoods of the form $\{\mathrm{Ex}(\bigcup_{j>n} P_j) : n \in \omega\}$ where $\{P_n : n \in \omega\}$ is a discrete sequence of non-empty open sets of a space X, then X is p-realcompact.*

PROOF. Let $Z = Z(f^\beta)$ be a zero-set of βX with $p \in X^* \cap Z$. We may assume that $p \in \upsilon X - X$ and so p has neighborhoods as in the hypothesis. Recursively pick $x_{j_n} \in P_{j_n} \cap f^{\leftarrow}[0, 1/n)$ for each $n \in \mathbb{N}$. Now $D = \{x_{j_n} : n \in \mathbb{N}\}$ is a closed discrete C-embedded subset of X and so D is not relatively pseudocompact in X. By Proposition 2.3 $\mathrm{cl}_{\beta X} D \not\subseteq \upsilon X$, but any point in $\mathrm{cl}_{\beta X} D - \upsilon X$ is in $Z - \upsilon X$.

A map $f: X \longrightarrow Y$ is *hyper-real* if $f^{\beta\rightarrow}(\beta X - \upsilon X) \subset \beta Y - \upsilon Y$. Hyper-real maps were introduced by Blair and are maps that carry realcompactness forward and pseudocompactness backward. Maps that are fiber countably compact and that carry zero-sets to closed sets (*e.g.* perfect maps) are hyper-real (see [13, 17.17, 17.19]). Not surprisingly hyper-real maps preserve p-realcompactness.

PROPOSITION 2.10. *If X is p-realcompact and $f: X \longrightarrow Y$ is a hyper-real surjective map, then Y is p-realcompact.*

PROOF. Let Z be a zero-set of βY that meets Y^*. Then $W = f^{\beta\leftarrow}(Z)$ is a zero-set of βX that meets X^*, and so there is a point $p \in W - \upsilon X$. Since f is hyper-real, then, $f^\beta(p) \in f^{\beta\rightarrow}(W) - \upsilon X \subset Z - \upsilon X$.

The next proposition shows that p-realcompact non-compact spaces have growths with rich structure.

PROPOSITION 2.11. *If X is p-realcompact and if Z is a zero-set of βX that meets X^*, then $X^* \cap Z$ contains a copy of $\beta\mathbb{N} - \mathbb{N}$ and hence has cardinality $\geq 2^{\mathfrak{c}}$.*

PROOF. This essentially follows from [8,9.5].

The question of whether the set-theoretic hypothesis MA + ¬ CH implies that every perfectly normal space is realcompact is still open as far as the authors know. It was a long-time conjecture of Blair that the question has an affirmative answer. (See [1] for a discussion of this question.) Of course, the question easily reduces to whether MA+¬ CH implies that perfectly normal spaces are c-realcompact and so the following proposition is of some interest.

PROPOSITION 2.12 (MA $+\neg$ CH). *Every perfectly normal space is hyperisocompact, hence p-realcompact.*

PROOF. This an easy consequence of Weiss's theorem in [14] that MA $+ \neg$ CH implies that every perfect countably compact space is compact.

3. **P'-spaces.** Before showing that locally compact p-realcompact spaces have growths that are P'-spaces, we need the following lemma. The proof given here is essentially that of [7,3.1].

LEMMA 3.0. *If X is locally compact, then every non-empty zero-set of βX that misses X has non-empty interior in X^*.*

PROOF. Let $Z = Z(f^\beta)$ be a non-empty zero-set of βX that misses X. Assume $f > 0$ on X. We can construct, recursively, sequences $\langle x_n : n \in \omega \rangle$ and $\langle f^{\leftarrow}(r_n, s_n) : n \in \omega \rangle$ such that $x_n \in f^{\leftarrow}(r_n, s_n) \subset f^{\leftarrow}(0, 1/n)$ and $f(x_n) \searrow 0$. Then for all n, we can pick a zero-set Z_n, a cozero-set P_n, and a compact set K_n with

$$x_n \subset Z_n \subset P_n \subset K_n \subset f^{\leftarrow}(r_n, s_n).$$

Let $P = \bigcup_{n\in\omega} P_n$. P is a cozero-set of X. We claim that $\emptyset \neq \mathrm{Ex}(P) \cap X^* \subset Z(f^\beta)$.

To show that $\emptyset \neq \mathrm{Ex}(P) \cap X^*$, let q be a limit point of $\{x_n : n \in \omega\}$ in βX. Since $q \in \mathrm{cl}_{\beta X} \bigcup_{n\in\omega} Z_n$, and $\bigcup_{n\in\omega} Z_n$ is a zero-set of X disjoint from $X - P$, $q \in \mathrm{Ex}(P) \cap X^*$. Now let $p \in \mathrm{Ex}(P) \cap X^*$ and suppose $f^\beta(p) = \epsilon > 0$. There is $k \in \omega$ with $s_k < \epsilon$. Then $p \notin \mathrm{cl}_{\beta X} \bigcup_{n\geq k} f^{\beta\leftarrow}(r_n, s_n)$ and so $p \notin \mathrm{cl}_{\beta X} \bigcup_{n\geq k} K_n$. But $\bigcup_{n<k} K_n$ is compact and so $p \notin \mathrm{cl}_{\beta X} \bigcup_{n\in\omega} K_n$, contradicting that $p \in \mathrm{Ex}(P)$. We conclude that $p \in Z(f^\beta)$.

To see that local compactness is necessary in the hypothesis of Lemma 3.0 we note (as did Fine and Gillman) that every zero-set of $\beta\mathbf{Q}$ that misses $\mathbf{Q}$ has empty interior in $\mathbf{Q}^*$

COROLLARY 3.1. *If X is locally compact and if Z is a zero-set of βX with $Z \not\subset \upsilon X$, then* $\mathrm{int}_{X^*}(X^* \cap Z) \neq \emptyset$.

PROOF. Let $p \in Z - \upsilon X$. There is $f \in C(\beta X)$ with $f(p) = 0$ and $f > 0$ on X. Then $\emptyset \neq Z(f) \cap Z \cap X^*$ while $Z(f) \cap Z \cap X = \emptyset$. By Lemma 3.0 $\emptyset \neq \mathrm{int}_{X^*}\big(Z(f) \cap Z\big) \subset \mathrm{int}_{X^*}(X^* \cap Z)$.

The following theorem is an immediate consequence of Corollary 3.1.

THEOREM 3.2. *If X is locally compact and p-realcompact, then X^* is a P'-space.*

PROPOSITION 3.3. *Let X be locally compact and nearly realcompact and let Z be a zero-set of βX. Then $Z \subset \upsilon X$ if and only if* $\mathrm{int}_{X^*}(Z - X) = \emptyset$.

PROOF. One direction is immediate from the definition of nearly realcompactness and the other is a consequence of Corollary 3.1.

COROLLARY 3.4. *If X is locally compact and nearly realcompact, then X^* is a P'-space if and only if X is p-realcompact.*

We give examples now to show that locally compact nearly realcompact spaces need not have P' remainders, that is, need not be p-realcompact.

EXAMPLE 3.5. This example of a locally compact, nearly realcompact space that is not p-realcompact is essentially due to Mrówka in [10]. Before describing it we give a bit of terminology and background.

Two infinite sets X and Y are said to be *almost disjoint* if $X \cap Y$ is finite. A family A of infinite sets is *almost disjoint* if each pair from A is almost disjoint. If A is maximal with respect to the property of being almost disjoint we say that A is a mad family. If A is an almost disjoint family of infinite subsets of ω, then there is a natural topology on the set $\Psi(A) = A \cup \omega$: Points of ω are isolated and if $a \in A$, then $\{\{a\} \cup (a - F) : F$ is a finite subset of $A\}$ is a neighborhood base at a. We call any such space a Ψ-space. Obviously Ψ-spaces are locally compact, and it is a fact that $\Psi(A)$ is pseudocompact if and only if A is maximal. See [8,5I] for more about Ψ-spaces.

In [10,3.11], Mrówka proves the existence of an Ψ-space $Y = \Psi(A)$ such that $\beta Y = Y \cup \{\infty\}$ is the one point compactification of Y. It is this space that we modify slightly for our example.

Let E be the set of even integers and let O be the set of odd integers. Let A_0 be a mad family on $\mathcal{P}(E)$ such that the Stone-Čech compactification of $\Psi(A_0)$ is the one point compactification of $\Psi(A_0)$. Let A_1 be a copy of A_0 on the odd integers and fix a countable $B \subset A_0$. Fix a bijection $f: (A_0 - B) \longrightarrow A_1$ and let X be the quotient space on $\Psi(A_0 - B) \cup \Psi(A_1)$ obtained by identifying each $a \in A_0 - B$ with $f(a)$. Let i_0 and i_1 be the inclusion maps from $\Psi(A_0 - B)$ and $\Psi(A_1)$ to X. We claim that X is nearly realcompact but not p-realcompact.

To see that X is nearly realcompact, fix any $p \in \beta X$ and fix any zero-set $Z \subseteq X$ in p. Let $Z_1 = A_1 \cap i_1^{\leftarrow}(Z)$. Now, Z_1 is a zero-set in $\Psi(A_1)$ and therefore is either finite (in which case Z is finite and $p \in X$), or co-countable. So suppose that it is co-countable, and fix an open set U containing Z. We claim that there is a $b \in B$ such that $U \cap b$ is infinite. To see this, let $Z_0 = A_0 \cap i_0^{\leftarrow}(Z)$. Z_0 is a co-countable subset of A_0 disjoint from B. Since A_0 is a mad family, $A_0 - Z_0$ is a mad family on the set $(\bigcup B) - U$. But no countable almost disjoint family can be maximal. Therefore there is a $b \in B$ such that $U \cap b$ is infinite. Since it is also clopen and discrete in X, we have that $\text{Ex}(U) - \upsilon X \neq \emptyset$.

To see that X is not p-realcompact, we first define a continuous function witnessing that $A = i_1^{\rightarrow}(A_1)$ is a zero-set in X. Let $B = \{b_k : k < \omega\}$ be an enumeration of B. Let $g(a) = 0$ for $a \in A$, $g(n) = 1/n$ for $n \in O$ and $g(n) = 1/k$ for $n \in B_k$. Note that the co-countable filter on A is a z-ultrafilter that converges to some $p \in \upsilon X$. The fact that A_0 and A_1 are mad families implies that $Z(g^{\beta}) = A \cup \{p\}$ and therefore $A \cup \{p\}$ is a zero set in βX witnessing that X is not p-realcompact.

Therefore this is an example of a locally compact and nearly realcompact space for which $\beta X - X$ is not P'. While this example is completely regular, it is not normal. Assuming CH, we can construct a normal example.

EXAMPLE 3.6. (CH) implies that there is a separable, normal, locally compact, nearly realcompact space that is not p-realcompact.

PROOF. Fix $A = \{a_\alpha : \alpha < \omega_1\}$, a copy of ω_1 disjoint from ω, and let $X = A \cup \omega$. Fix a partition of $\omega = \bigcup_{n<\omega} B_n$ such that each B_n is infinite. We define a topology τ on X such that

(1) (X, τ) is locally compact, locally countable, first countable and 0-dimensional.
(2) ω consists of isolated points.
(3) $A \cup \bigcup_{k>n} B_k$ is open for each n, and hence if $f : X \longrightarrow [0, 1]$ is defined by $f^{\leftarrow}\{0\} = A$ and $f^{\leftarrow}\{1/n\} = B_n$ for each n, then f is continuous.
(4) A is homeomorphic to ω_1 with the order topology.
(5) For each infinite $b \subseteq \omega$, if $b \cap B_n \neq \emptyset$ for infinitely many $n \in \omega$, then b has an accumulation point in A.
(6) For each open U containing a co-countable subset $A' \subseteq A$, there is an open set V such that $A' \subseteq V \subseteq U$ and $U - V$ is infinite.

The local compactness and (4) imply that any such space is normal. Also it is nearly realcompact but not p-realcompact: The fact that the co-countable filter on A is a z-ultrafilter follows from (3) and (4). If $p \in \upsilon X$ is its limit, then (5) implies that $A \cup \{p\}$ is a zero-set witnessing that the space is not p-realcompact. Normality along with (6) imply nearly realcompactness. The proofs are analogous to the same results for the first example.

Before we present the construction we need some notation. Let $A_\alpha = \{a_\beta : \beta \leq \alpha\}$. We will call a countable set $b \subseteq \omega$ a transversal sequence if b is infinite and $b \cap B_n \leq 1$ for each $n < \omega$. Using CH we enumerate all transversal sequences as $\{b_\alpha : \alpha < \omega_1\}$. The example is defined recursively by constructing topologies τ_α on $X_\alpha = A_\alpha \cup \omega$ satisfying the following inductive hypotheses:

(a) $\beta < \alpha$ implies that τ_α is a conservative extension of τ_β (*i.e.*, τ_β coincides with the subspace topology on X_β from τ_α, and X_β is open in X_α).
(b) (X_α, τ_α) is locally compact, 0-dimensional, locally countable and first countable.
(c) b_β has an accumulation point in A_α for each $\beta < \alpha$.
(d) ω consists of isolated points in X_α.
(e) $A_\alpha \cup \bigcup_{k>n} B_k$ is open in X_α.

At limit stages $\alpha < \omega_1$ let τ'_α be the topology on $X_\alpha - \{a_\alpha\}$ generated by taking $\bigcup_{\beta<\alpha} \tau_\beta$ as a base. Note that τ'_α is a conservative extension of each τ_β. To define τ_α it suffices to define a local neighborhood base at a_α. Fix an increasing sequence $(\alpha_n)_{n\in\omega}$ with supremum α. For each n, the set $(a_{\alpha_n}, a_{\alpha_{n+1}}]$ is compact with respect to τ'_α. Therefore we can fix a compact open neighborhood $V_n \supseteq (a_{\alpha_n}, a_{\alpha_{n+1}}]$, so that $V_n \cap \omega \subseteq \bigcup_{k>n} B_k$. We also demand that the V_n's are pairwise disjoint. Let $U_n = \{a_\alpha\} \cup \bigcup_{k>n} V_k$ for each $n \in \omega$. If we let τ_α be generated by $\tau'_\alpha \cup \{U_n : n \in \omega\}$, then the inductive hypotheses are preserved.

At successor stages $\alpha = \beta + 1$ we consider the transversal sequence b_β. If it already has an accumulation point in A_β then we leave a_α isolated. Otherwise we extend τ_β to τ_α by letting $U_F = \{a_\alpha\} \cup b_\beta \setminus F$ be open for each finite set $F \subseteq b_\beta$. This makes b_β converge to a_α in (X_α, τ_α). Clearly the inductive hypotheses are preserved, thus completing the construction.

We need to check that the example satisfies the requirements (1)–(6). The inductive hypotheses imply (1), (2), (3), and (5). Hypothesis (4) follows easily from the construction. To see that (6) holds, fix an open set U containing a co-countable subset C of A. And suppose that $U \cap B_n$ were finite for each $n \in \omega$. Let $(a_{\alpha_n})_{n\in\omega}$ be an enumeration of the complement of C. For each n, fix a compact neighborhood U_n of a_{α_n} such that $U_n \cap \omega \subseteq \bigcup_{k>n} B_k$. Note also that, by compactness, U_n meets each of the B_k in a finite set. Let $V = U \cup \bigcup_{n\in\omega} U_n$. Then V is an open set containing A such that $V \cap B_n$ is finite for each n. This contradicts requirement (5). Therefore there is an n such that $U \cap B_n$ is infinite. Then $V = U - \bigcup_{k\leq n} B_k$ is an open set, $C \subseteq V \subseteq U$ and $U - V$ is infinite. This completes the construction.

We do not know whether there is an example in ZFC of a normal locally compact nearly realcompact space that is not p-realcompact.

We turn now to the question of what local conditions on a zero-set $Z(f)$ of a locally compact nearly realcompact space X will guarantee that $\mathrm{int}_{X^*}\big(X^* \cap Z(f^\beta)\big) = \emptyset$. We say that $f \in C(X)$ is a *well separated function* in X if whenever $Z(f) \subset P$ where P is a cozero-set of X, there is $n \in \mathbb{N}$ with $f^{\leftarrow}(-1/n, 1/n) \subset P$. We will use the following result.

PROPOSITION 3.7 [12,4.1]. *Let X be a space and let $f \in C^*(X)$. Then $\mathrm{cl}_{\beta X} Z(f) = Z(f^\beta)$ if and only if f is well separated in X.*

PROPOSITION 3.8. *For any space X, the following are equivalent.*

(1) X is p-realcompact

(2) Every relatively pseudocompact zero-set of a well separated function is compact.

PROOF. (1) ⇒ (2). Let $Z(f)$ be a relatively pseudocompact and let f be a well separated function in X. By Proposition 2.3 and 3.7, $Z(f^\beta) = \mathrm{cl}_{\beta X} Z(f) \subset \upsilon X$. Since X is p-realcompact, $Z(f^\beta) \cap X^* = \emptyset$ which implies that $Z(f)$ is compact.

(2) ⇒ (1). Let (1) be false and let $\emptyset \neq Z(f^\beta) \cap X^* \subset \upsilon X$. By Proposition 2.2, $\mathrm{cl}_{\beta X} Z(f) = Z(f^\beta)$ and so by Proposition 3.7 and 2.3, $Z(f)$ is relatively pseudocompact and f is well separated. Then (2) is false.

PROPOSITION 3.9. *Let X be locally compact and let $Z(f)$ be a zero-set of X. Then (1) ⇒ (2) below and if, in addition, X is nearly realcompact, then (1) and (2) are equivalent.*

(1) $\mathrm{int}_{X^*}\big(X^* \cap Z(f^\beta)\big) = \emptyset$.

(2) $Z(f)$ is relatively pseudocompact, and f is well separated in X.

PROOF. (1) ⇒ (2). If $Z(f)$ is not relatively pseudocompact, then by Proposition 2.3 $\mathrm{cl}_{\beta X} Z(f) \not\subseteq \upsilon X$ and so by Corollary 3.1 $\mathrm{int}_{X^*}\big(X^* \cap Z(f^\beta)\big) \neq \emptyset$. Suppose next that f is not well separated. Then by Proposition 3.7, $\mathrm{cl}_{\beta X} Z(f) \neq Z(f^\beta)$ and so by Proposition 2.2, $Z(f^\beta) \not\subseteq \upsilon X$. Then by Corollary 3.1, (1) is false.

(2) ⇒ (1). Assume now that X is nearly realcompact. Then this implication follows from Proposition 3.7, 2.2 and 3.3.

COROLLARY 3.10. *If X is locally compact, then (2) $\Rightarrow$ (1). If, in addition, X is nearly realcompact, then (1) and (2) are equivalent.*

(1) X^* is P'.

(2) Every relatively pseudocompact zero-set of a well separated function is compact.

We remark that locally compact spaces with P' growths need not have any weak realcompactness properties. Any almost compact non-compact space (*e.g.* ω_1) has a P' growth but has no weak realcompactness properties. Thus the hypothesis that X is nearly realcompact cannot be dropped in Proposition 3.9 or Corollary 3.10.

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Canad. Math. Bull. Vol. **39** (4), 1996 pp. 486–498

THE WALLACE PROBLEM: A COUNTEREXAMPLE FROM MA$_{\text{COUNTABLE}}$ AND p-COMPACTNESS

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ABSTRACT. We construct, under MA$_{\text{countable}}$, a countably compact topological subsemigroup of $T^{\mathfrak{c}}$ which is not a group, hence a counterexample for the Wallace problem. We also show that there is no p-compact counterexample for the Wallace problem, answering a question of D. Grant. Finally, we show that—in some sense—our counterexample for the Wallace problem constructed under MA$_{\text{countable}}$ cannot be done in ZFC.

Introduction. We call the Wallace problem the question 3L.1 in [1]: *Is every countably compact topological semigroup with two-sided cancelation a topological group*? It was asked by Wallace in 1953 at the annual meeting of the American Mathematical Society in Baltimore, Maryland.

Robbie and Svetlichny showed recently under CH [10] that there is a counterexample for the Wallace problem using Tkachenko's group [11]. We have shown in [12] that there is a countably compact free abelian group without non-trivial convergent sequences under MA (σ-centered), modifying van Douwen's example in [2]. An immediate corollary of the existence of such a group is the existence of a counterexample for the Wallace problem. We show here that there is a counterexample to Wallace's problem under MA$_{\text{countable}}$ directly, without finding such a group as above. We remind the reader that there is no free abelian group whose ω-th power is countably compact [12], hence one cannot have this example as a subsemigroup of a free abelian group, since our example contains an ω-bounded subgroup.

We note that what makes the solution presented here simpler is the fact that we were able to give a solution to Wallace's problem without having to deal with the product not being countably compact (in [12], our simplest example would contain two counterexamples for the Wallace problem whose product is not countably compact, by using van Douwen's argument of finding two countably compact groups whose product is not countably compact from a countably compact group without non-trivial convergent sequences).

We can get from MA$_{\text{countable}}$ a counterexample for the Wallace problem whose square is not countably compact. The construction is longer and more complicated and it is part of our Ph.D thesis.

This work has been partially supported by CNPq and University of São Paulo—São Paulo, Brazil.

Received by the editors March 17, 1995; revised January 18, 1996.

AMS subject classification: Primary: 22A99, 54A35; secondary: 03E50.

Key words and phrases: Wallace problem, MA$_{\text{countable}}$, p-compactness, countable compactness, topological semigroups.

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In the first section, we give a self-contained proof of the existence of the counterexample for the Wallace problem, applying a strong form of the Baire Category Theorem that is equivalent to $\mathrm{MA}_{\mathrm{countable}}$.

In the second section, we show how to modify the construction to obtain other non-isomorphic counterexamples for the Wallace problem which are subsemigroups of $T^{\mathfrak{c}}$.

In the third section we show that there are no regular p-compact counterexamples for the Wallace problem, answering a question from [4]. We apply it to show that certain type of counterexamples for the Wallace problem are independent of ZFC.

The first submitted version of this work is part of the third chapter of my thesis [14]. I would like to thank my supervisor Prof. Steve Watson, for his continuing support during my Ph.D studies at York University.

1. A counterexample for the Wallace problem from $\mathrm{MA}_{\mathrm{countable}}$.

We will first give a general idea of the construction, then we will give the details. We recall the following definitions:

DEFINITION 1.1. A topological semigroup is a semigroup whose addition is continuous.

DEFINITION 1.2. A space is ω-bounded if the closure of any countable subset is compact.

We remind the reader that $\mathrm{MA}_{\mathrm{countable}}$ is Martin's Axiom restricted to countable posets. This axiom is equivalent to a strong form of the Baire Category Theorem:

(#) The circle T is not the union of fewer than continuum many closed nowhere dense sets.

In this work we will only use (#), so no familiarity with partial orders and generic sets will be required.

For someone who is more comfortable with the Continuum Hypothesis but unfamiliar with Martin's Axiom, note that under the Continuum Hypothesis, (#) is just the usual Baire Category Theorem. We recall that MA implies MA (σ-centered) and MA(σ-centered) implies $\mathrm{MA}_{\mathrm{countable}}$ and that the reverse of either implication is not true. A proof of the consistency of MA can be found in Kunen's book [6]. More on MA and its variations can be found in [16].

1.1. *General framework.*

We remind the reader that Robbie and Svetlichny's counterexample was a semigroup that was not a group. The same is true for our counterexample we constructed under MA (σ-centered). In fact, if one is looking for a Tychonoff counterexample for the Wallace problem, that must be the case, since a Tychonoff countably compact group which has continuous addition is a topological group (Reznichenko shows in [9] that it is also true for pseudocompact instead of countably compact).

The semigroup S we construct here will be the (algebraic) direct sum of an ω-bounded group G and a semigroup generated by an element x. As in the previous examples of the

Wallace problem, the semigroup will be a subsemigroup of $T^{\mathfrak{c}}$, where T is the circle with the usual group topology.

More precisely, our semigroup S will be the subsemigroup $\{nx+g : n \in \omega, g \in G\}$.

Clearly S is a both-sided cancellative topological semigroup, since S is a subsemigroup of a topological group.

To make S not a group, we will make sure that x does not have an inverse in S. For that, it is enough that

$$(*) \qquad \text{for all } n \in \omega, \text{ if } nx \in G \text{ then } n = 0.$$

In fact, suppose there exist $n \in \omega$ and $g \in G$ such that $(nx + g) + x = 0$. Then $(n + 1)x = -g \in G$. Therefore, by $(*)$, $n + 1 = 0$, contradicting that $n \in \omega$.

Countable compactness of S will be achieved using Hart-van Mill's idea from [5], that is, we will make sure that

$$(**) \qquad \text{every infinite subset of } \{nx : n \in \omega\} \text{ has an accumulation point in } G.$$

As we will see in Lemma 1.1, this will be enough since G is ω-bounded.

We note that unlike in [5], our ω-bounded group is fixed from the beginning and might be definable in ZFC.

Instead of using an ω-independent family on $\mathfrak{c}$ as in [5], we will use a proper σ-ideal on $\mathfrak{c}$ containing all subsets of size less than $\mathfrak{c}$. We also would like to mention that the ω-bounded group in [5] has size $\mathfrak{c}$, but ours will have size at least $\sup\{2^\alpha : \alpha < \mathfrak{c}\}$.

Before giving more details about the construction, we will need some basic definitions:

DEFINITION 1.3. An ultrafilter p on ω is a non-empty family of subsets of ω satisfying the following:

(1) $\emptyset \notin p$.
(2) The intersection of finitely many elements of p is an element of p.
(3) Every subset of ω containing an element of p is an element of p.
(4) For each $A \subseteq \omega$, either $A \in \omega$ or $\omega \setminus A \in \omega$.

A free ultrafilter p on ω is an ultrafilter which does not contain finite subsets of ω.

Let p be a free ultrafilter on ω and X a topological Hausdorff space.

DEFINITION 1.4. An element $x \in X$ is a p-limit of a sequence $\{x_n : n \in \omega\}$ if for every open neighbourhood U of x, $\{n \in \omega : x_n \in U\} \in p$.

Note that by Hausdorffness, the p-limit of a sequence is unique. We will denote this unique point (if it exists) as $p\text{-}\lim\{x_n : n \in \omega\}$. We remind the reader that the p-limits have nice properties of usual limits of a converging sequence, for instance, for a topological semigroup, the sum of p-limits of two sequences equals the p-limit of the sum of those sequences.

We will show now that $(**)$ is sufficient to have S countably compact:

LEMMA 1.1. *Let S be the semigroup generated by x and G, where G is ω-bounded. If* $(**)$ *is satisfied then S is countably compact.*

PROOF OF LEMMA 1.1. The proof of this lemma is as in [5], but we will give it here for completeness sake.

Let $\{y_k : k \in \omega\}$ be an infinite subset of S. Then for each $k \in \omega$ we can find $n_k \in \omega$ and $g_k \in G$ such that $y_k = n_k x + g_k$. We have two cases to consider:

(i) There is an infinite subset A of ω such that $n_k \neq n_l$ if $k, l \in A$ and $k \neq l$.

From $(**)$, $\{n_k x : k \in A\}$ has an accumulation point y in G. Let p be a free ultrafilter over A such that y is a p-limit of $\{n_k x : k \in A\}$. Since G is ω-bounded, there exists $g \in G$ such that g is p-limit of $\{g_k : k \in A\}$. Clearly $y + g \in G \subseteq S$ is a accumulation point of $\{y_n : n \in \omega\}$ and we are done in this case.

(ii) NOT CASE (i). In this case, we can find an infinite subset B of ω and $n' \in \omega$ such that for each $k \in B$ we have $n_k = n'$. Since G is countably compact, there exist $g \in G$ such that g is an accumulation point of $\{g_k : k \in B\}$. Clearly, $n'x + g \in S$ is an accumulation point of $\{y_n : n \in \omega\}$ hence we are also done in this case. ∎

From our discussion above, in order to get a counterexample for the Wallace problem, it suffices to find $x \in T^{\mathfrak{c}}$ and G an ω-bounded subgroup of $T^{\mathfrak{c}}$ satisfying properties $(*)$ and $(**)$.

1.2. *More details.* We start by telling exactly what G will look like.

DEFINITION 1.5. Given $y \in T^{\mathfrak{c}}$, $\operatorname{supp} y = \{\alpha < \mathfrak{c} : y(\alpha) \neq 0\}$. Given a σ-ideal I, on $\mathfrak{c}$, the group generated by I is the group $G_I = \{y \in T^{\mathfrak{c}} : \operatorname{supp} y \in I\}$. When it is clear which I we are using, we will denote it simply by G.

We remind that a σ-ideal I on $\mathfrak{c}$ is a family of subsets of $\mathfrak{c}$, satisfying the following:

(1) $\emptyset \in I$,
(2) every subset of an element of I is an element of I and
(3) the union of countable many elements of I is an element of I.

We have now defined all we need to state our goal in this section:

EXAMPLE 1.1 ($\mathrm{MA}_{\mathrm{countable}}$). *There exists $x \in T^{\mathfrak{c}}$ such that for each proper σ-ideal I containing all bounded subsets of $\mathfrak{c}$, the semigroup generated by x and G_I is a counterexample for the Wallace problem.*

We will shortly see the reason to restrict ourselves to proper σ-ideals containing all bounded subsets of $\mathfrak{c}$.

We will denote by 0 the neutral element of T or T^{ξ} for $\xi \leq \mathfrak{c}$ and this will be clear by the context.

How will we get $(*)$ and $(**)$?

We already know what the ω-bounded group G will be, but we need yet to say how we are going to get x. At stage $\gamma + 1, \gamma < \mathfrak{c}$, we will define $x(\gamma)$.

For $(*)$ and $(**)$ to be satisfied, we will need a couple of inductive hypothesis that must be satisfied during the construction of x.

To show $(*)$, it suffices to show that for each $n \in \omega$, and for every $y \in G$, there is a β in $\mathfrak{c}$ such that $nx(\beta) \neq y(\beta)$. In fact we will have $\beta \in \mathfrak{c}$ such that $y(\beta) = 0 \neq nx(\beta)$. This can be done by making $\operatorname{supp} nx = \mathfrak{c}$, as $\operatorname{supp} y$ is a proper subset of $\mathfrak{c}$.

In order to explain the inductive hypothesis needed to get $(**)$, we have to fix first an enumeration $\{E_\alpha : \alpha < \mathfrak{c}\} = [\omega]^\omega$ of all infinite subsets of ω.

This will be used to code every infinite subset of $\{nx : n \in \omega\}$, that is, for every subset A of $\{nx : n \in \omega\}$ there exists $\alpha < \mathfrak{c}$ such that $A = \{nx : n \in E_\alpha\}$.

To show $(**)$, we need to guarantee an accumulation point for each infinite subset of $\{nx : n \in \omega\}$. Roughly speaking, during the inductive process we will promise to the countable subsets of $\{nx : n \in \omega\}$ an accumulation point.

As we said earlier, we only will know completely x and consequently $\{nx : n \in \omega\}$ in the end of the construction, but we know more about the sequence as we know more about x because of the coding we mentioned earlier. More precisely, at stage β, we will know $x{\restriction}\beta$, therefore we can code $\{nx{\restriction}\beta : n \in E_\beta\}$. We will assign a function $g_\beta \in T^{\mathfrak{c}}$ such that $\operatorname{supp} g_\beta$ is bounded and $g_\beta{\restriction}\beta$ is accumulation point of $\{nx{\restriction}\beta : n \in E_\beta\}$. From then on, we will make sure that for each $\alpha > \beta$, the point $g_\beta{\restriction}\alpha$ is accumulation point of $\{nx{\restriction}\alpha : n \in E_\beta\}$. In particular, at the end of the construction, $g_\beta \in G$ will be an accumulation point of $\{nx : n \in E_\beta\}$.

As every subset of $\{nx : n \in \omega\}$ is coded by some E_β, every subset of $\{nx : n \in \omega\}$ will have an accumulation point in G and $(**)$ will be satisfied.

Note that the x we find works for any group generated by a proper σ-ideal containing all bounded subsets of $\mathfrak{c}$, as $(*)$ and $(**)$ will be simultaneously satisfied for x and any of the groups just described.

Summarizing what we have done so far:

LEMMA 1.2. *Suppose that there exist* $\{x_\alpha : \alpha \leq \mathfrak{c}\}$ *and* $\{g_\beta : \beta < \mathfrak{c}\} \subseteq \{g \in T^{\mathfrak{c}} :$ $\operatorname{supp} g$ *is bounded in* $\mathfrak{c}\}$ *such that the following are satisfied:*

(0) for each $\beta \leq \mathfrak{c}$ *we have* $x_\beta \in T^\beta$.

(1) if $\beta < \alpha \leq \mathfrak{c}$ *then for each* $n \in \mathbb{N}$ *we have* $nx_\alpha(\beta) \neq 0$ *and* $x_\beta \subseteq x_\alpha$.

(2) if $\beta < \alpha \leq \mathfrak{c}$ *then* $g_\beta{\restriction}\alpha$ *is an accumulation point of* $\{nx_\alpha : n \in E_\beta\}$.

Then $(*)$ *and* $(**)$ *are satisfied for* $x = x_{\mathfrak{c}}$ *and any* G_I, *where* I *is a proper* σ*-ideal containing all bounded subsets of* $\mathfrak{c}$. *Therefore the semigroup generated by* x *and each* G_I *is a counterexample for the Wallace problem.*

Therefore, to get Example 1.1, it suffices to show that we can produce $\{x_\alpha : \alpha \leq \mathfrak{c}\}$ and $\{g_\beta : \beta < \mathfrak{c}\}$. For some σ-ideals, such families do not exist in ZFC. In fact, in Section 3, we will see that there is a model in which for the σ-ideal I of all bounded subsets of $\mathfrak{c}$, there is no x such that the semigroup generated by x and G_I is a counterexample for the Wallace problem.

1.3. *The induction.* We will show now that using (#), the conditions in Lemma 1.2 can be satisfied.

We will be done with the construction of Example 1.1 by proving the following:

LEMMA 1.3 ($\mathrm{MA}_{\mathrm{countable}}$). *There exist* $\{x_\alpha : \alpha \leq \mathfrak{c}\}$ *and* $\{g_\beta : \beta < \mathfrak{c}\}$ *satisfying conditions (0)–(2) from Lemma 1.2.*

PROOF OF LEMMA 1.3. These families are constructed by induction.

How g_γ is chosen: At stage $\gamma + 1$, we will fix y_γ any accumulation point of $\{nx_\gamma : n \in E_\gamma\}$ and define $g_\gamma = y_\gamma \cup 0\restriction[\gamma, \mathfrak{c})$.

How to define x_α's.

We will do it in cases, we note that we will use MA$_{\text{countable}}$ (in fact (#)) only in the successor stage:

CASE 1. $\alpha = 1$. Let $x_1 \in T^1$ be any function such that for each $n \in \mathbb{N}$ we have $nx_1(0) \neq 0$.

CASE 2. α limit. Let $x_\alpha = \bigcup_{\beta<\alpha} x_\beta$.

Clearly the inductive hypothesis are satisfied in both cases so we will now show how to deal with the successor case.

CASE 3. $\alpha = \gamma + 1$. As mentioned before we define $g_\gamma \in T^{\mathfrak{c}}$ such that $g_\gamma\restriction\gamma$ is an accumulation point of $\{nx_\gamma : n \in E_\gamma\}$.

For every $\beta \leq \gamma$, $F \in [\gamma]^{<\omega}$ and $k \in \omega$ define

$$S(\beta, F, k) = \left\{n \in E_\beta : \forall\eta \in F |nx_\alpha(\eta) - g_\beta(\eta)| < \frac{1}{k+1}\right\}.$$

Note that $S(\beta, F, k)$ is infinite. Now, we have to choose $x_\alpha(\gamma)$ which fulfill the promises.

Note that it suffices that for all β, F and k as above the set $\{n \in S(\beta, F, k) : |nx_\alpha(\gamma) - g_\beta(\gamma)| < \frac{1}{k+1}\}$ is infinite and that $nx_\alpha(\gamma) \neq 0$ for all $n \in \mathbb{N}$.

In order to apply (#) we define now fewer than $\mathfrak{c}$ many dense open sets:

(a) For each $n \in \mathbb{N}$ let $O_n = \{a \in T : na \neq 0\}$.

(b) For each $\beta \leq \gamma$, $F \in [\gamma]^{<\omega}$ and $k, m \in \omega$, let

$$O(\beta, F, k, m) = \left\{a \in T : \exists n \in S(\beta, F, k)\left[n \geq m \text{ and } |na - g_\beta(\gamma)| < \frac{1}{k+1}\right]\right\}$$

CLAIM 1. *The sets defined above are dense open in T.*

PROOF OF THE CLAIM. (a) Each O_n has finite complement and hence is open and dense in T.

(b) Fix $O(\beta, F, k, m)$.

We will show first that this set is open. Let $b \in O(\beta, F, k, m)$. By definition, there exist $n \in S(\beta, F, k)$ such that $n \geq m$ and $|nb - g_\beta(\gamma)| < \frac{1}{k+1}$. By continuity, clearly there exist an open set W containing b such that for all $a \in W$ we have $|na - g_\beta(\gamma)| < \frac{1}{k+1}$. Therefore $W \subseteq O(\beta, F, k, m)$ and we are done.

We will show now that $O(\beta, F, k, m)$ is dense in T. Let V be any open subset of T. Let $n \in S(\beta, F, k)$ such that $n \geq m$ and $nV = T$. Let $a \in V$ such that $|na - g_\beta(\gamma)| < \frac{1}{k+1}$. Then $a \in O(\beta, F, k, m)$ as well, therefore, $V \cap O(\beta, F, k, m) \neq \emptyset$ so $O(\beta, F, k, m)$ is dense. ■

By (#), $\bigcap_{\beta,F,k,m} O(\beta, F, k, m) \cap \bigcap_{n\in\mathbb{N}} O_n \neq \emptyset$.

Let a be any element of $\bigcap_{\beta,F,k,m} O(\beta, F, k, m) \cap \bigcap_{n\in\mathbb{N}} O_n$. To end the proof of the Lemma 1.3, it suffices to show the following

CLAIM 2. *$x_\alpha = x_\gamma \cup \{\langle\gamma, a\rangle\}$ satisfies the inductive hypothesis.*

Proof of the Claim. Clearly (0) is satisfied by x_α. Also it is easy to see that (1) is satisfied, since $x_\alpha(\gamma) = a \in O_n = T \setminus \{b \in T : nb = 0\}$. Let us show now that (2) is satisfied for x_α and we will be done. As noted before, to show that $g_\beta \restriction \alpha$ is an accumulation point of $\{nx_\alpha : n \in E_\beta\}$, it suffices to show that for every $F \in [\alpha]^{<\omega}$ and $k \in \omega$ we have that the set $\{l \in S(\beta, F, k) : |lx_\alpha(\gamma) - g_\beta(\gamma)| < \frac{1}{k+1}\} = \{l \in E_\beta : (\forall \eta \in F \cup \{\gamma\}) |lx_\alpha(\eta) - g_\beta(\eta)| < \frac{1}{k+1}\}$ is infinite.

Let $m \in \mathbb{N}$. By construction, $x_\alpha(\gamma) \in O(\beta, F \setminus \{\gamma\}, k, m)$, that is, there exists $n \in S(\beta, F \setminus \{\gamma\}, k)$ such that $n \geq m$ and $|nx_\alpha(\gamma) - g_\beta(\gamma)| < \frac{1}{k+1}$. Therefore $\{l \in S(\beta, F, k) : |lx_\alpha(\gamma) - g_\beta(\gamma)| < \frac{1}{k+1}\}$ is unbounded in ω and we are done. ■

Note. We could have made $\{nx : n \in \omega\}$ dense in $T^{\mathfrak{c}}$ by adding some new dense open sets at successor stages. ■

Remarks. It is a short proof (in ZFC) to show that

every countably compact torsion free group such that every countable subset
(+) of it has closure of size at least $\mathfrak{c}$ contains a semigroup that is a
counterexample for the Wallace problem.

Our example has convergent sequences as well as elements of finite order hence it could not be obtained by Robbie-Svetlichny's approach, since they relied on (+) to obtain their counterexample under CH.

We remind that every countably compact free abelian group without convergent sequences satisfies (+).

In [12] we showed that every countably compact free abelian group without non-trivial convergent sequences contains at least $|\alpha| + \aleph_0$ non-isomorphic counterexamples for the Wallace problem, where $\mathfrak{c} = \aleph_\alpha$.

We also showed that under MA (σ-centered), there exist a semigroup and $\sup\{\kappa < \mathfrak{c} : \kappa \text{ cardinal}\}$ non-homeomorphic topologies that make it a separable counterexample for the Wallace problem whose square is not countably compact (also, under MA (σ-centered), there are $\sup\{\kappa < \mathfrak{c} : \kappa \text{ cardinal}\}$ non-homeomorphic topologies that make the free abelian group generated by $\mathfrak{c}$ many elements a countably compact separable topological group whose square is not countably compact).

2. Some non-isomorphic counterexamples inside $T^{\mathfrak{c}}$.

We will show now how to get some other non-isomorphic counterexamples inside $T^{\mathfrak{c}}$ by choosing some special σ-ideals.

Definition 2.1. A topological space X is κ-bounded if every subset of X of size less or equal to κ has compact closure.

Notation. The cofinality of κ will be denoted by $\mathrm{cf}(\kappa)$.

Let z be the unique non-zero element of T such that $z + z = 0$. Given a subset A of $\mathfrak{c}$, we will denote by χ_A the function from $\mathfrak{c}$ into $\{0, z\}$ such that $\chi_A(a) = z$ if and only if $a \in A$. We will call it here the characteristic function of A.

EXAMPLE 2.1 ($\mathrm{MA}_{\mathrm{countable}}$). *Suppose* $\kappa < \mathrm{cf}(\mathfrak{c})$ *is an infinite cardinal. Then there exist* $x \in T^{\mathfrak{c}}$ *and* $G_\kappa \subseteq T^{\mathfrak{c}}$ *that is* κ*-bounded but not* κ^+*-bounded such that the semigroup generated by* x *and* G_κ *is a counterexample for the Wallace problem.*

In order to get a κ-bounded group, we will choose a proper κ-ideal. For this, we will use the fact that $\kappa < \mathrm{cf}(\mathfrak{c})$.

THE CHANGES. As mentioned above we choose some special ideals. Fix $\mathcal{A} = \{A_\alpha : \alpha < \mathfrak{c}\}$ such that for all $\alpha < \mathfrak{c}$ A_α is a subset of $\mathfrak{c}$ of size $\mathfrak{c}$ and if $\alpha \neq \beta$ then $A_\alpha \cap A_\beta = \emptyset$.

Let I_κ be the κ-ideal generated by $\mathcal{A} \cup \{\alpha : \alpha < \mathfrak{c}\}$. (Since $\kappa < \mathrm{cf}(\mathfrak{c})$, I_κ is proper). Let G_κ be the group generated by I_κ.

Clearly G_κ is κ-bounded. To check that G_κ is not κ^+-bounded, first note that $A_\alpha \in I_\kappa$ and $\bigcup_{\alpha<\kappa^+} A_\alpha \notin I_\kappa$.

Then $\{\chi_{\bigcup_{\alpha\in F} A_\alpha} : F \in [\kappa^+]^{<\omega}\} \in [G_\kappa]^{\kappa^+}$, $\chi_{\bigcup_{\alpha<\kappa^+} A_\alpha} \in \{\overline{\chi_{\bigcup_{\alpha\in F}} A_\alpha : F \in [\kappa^+]^{<\omega}}\}$, but $\chi_{\bigcup_{\alpha<\kappa^+} A_\alpha} \notin G_\kappa$. Hence G_κ is not κ^+-bounded. Since I_κ satisfies the conditions from Example 1.1, we are done. ∎

REMARK. I would like to thank the referee for pointing out that an enumeration we were using to obtain a condition weaker than $(*)$ was not necessary and for suggesting the sets O_n to obtain $(*)$. From that we could use more general σ-ideals as well (our σ-ideals used to be the ones generated by $\mathcal{A} \cup \{\alpha : \alpha < \mathfrak{c}\}$). Previously, because of the enumeration, the example above was not a particular case of the example in Section 1.

We note that we could use O_n for T but not for 2, since $\{0\}$ is open in 2. In [13], we constructed a subgroup of $2^{\mathfrak{c}}$ whose square is countably compact but whose cube is not. In that case, it seems we need our original condition related to $(*)$ using the enumeration.

COROLLARY 2.1 ($\mathrm{MA}_{\mathrm{countable}}$). $T^{\mathfrak{c}}$ *contains at least* $|\gamma|$ *many non-isomorphic counterexamples for the Wallace problem, where* $\mathrm{cf}(\mathfrak{c}) = \aleph_\gamma$.

PROOF OF THE COROLLARY. From Example 2.1, for all $\alpha < \gamma$ there exist x and $G_{\aleph_\alpha}$ such that the semigroup S_α generated is a counterexample for the Wallace problem. We claim that for $\alpha < \beta < \gamma$, the semigroups S_α and S_β are not isomorphic. For this, it suffices to notice that the set of elements of S_α that have inverse is $G_{\aleph_\alpha}$ and that $G_{\aleph_\alpha}$ is not homeomorphic to $G_{\aleph_\beta}$. ∎

REMARKS. All the semigroups we have constructed here so far have size at least $\sup\{2^\lambda : \lambda < \mathfrak{c}\}$, as it contains all elements of $T^{\mathfrak{c}}$ whose support is bounded. However, using closing-off arguments one can get a counterexample for the Wallace problem of size $\mathfrak{c}$.

3. Further results and limitations.

As we discussed in the introduction, the first counterexample for the Wallace problem was only obtained in 1994. Some authors have proved before that a counterexample for the Wallace problem cannot have certain properties. In 1957, Ellis [3] showed that a locally compact group whose addition is assumed only to be separately continuous is a topological group, in particular a counterexample for the Wallace problem cannot be algebraically a group and locally compact.

We recall that a locally compact topological semigroup is not necessarily a group, in fact, $[0, 1)$ with the usual multiplication is an example. In 1972, Mukherjea and Tserpes [7] showed that there is no first countable counterexample for the Wallace problem.

Pfister [8] showed that every (locally) countably compact regular paratopological group (addition is continuous, but the inversion is not necessarily continuous) is a topological group, in particular, a counterexample for the Wallace problem cannot be algebraically a group.

Grant [4] gave some other properties that a counterexample for the Wallace problem cannot have, among them, sequential compactness. He also mentioned that it was known that there is no ω-bounded counterexample for the Wallace problem.

3.1. *p-compactness.* We recall the following:

DEFINITION 3.1. A p-compact space is a Hausdorff space such that every sequence has a p-limit.

We remind that every ω-bounded space is p-compact for every free ultrafilter p on ω. More on p-limits and p-compact spaces can be found in [15].

The fact that there is no sequentially compact or ω-bounded counterexample for the Wallace problem motivated the following question asked in [4]:

QUESTION 3. *Is every p-compact* ($T_{3\frac{1}{2}}$), *cancellative topological semigroup a topological group?*

We have shown the following:

THEOREM 3.1. *Let p be a free ultrafilter. Then every p-compact both-sided cancellative semigroup is a group.*

From Theorem 3.1 and Pfister's result mentioned above, we answer Question 3. We note that one can modify the proof of Theorem 3.1 below to show that every sequentially compact both-sided cancellative semigroup is a group.

PROOF OF THEOREM 3.1 . Suppose there exist an S that is p-compact both-sided cancellative semigroup but is not a group. Without loss of generality we can assume that S has the neutral element 0. In fact, if S does not have the neutral element then let $0 \notin S$ and define $S \cup \{0\}$, where $\{0\}$ is an open subset and we consider the topological sum of S and $\{0\}$.

Note that we still would have a p-compact both-sided cancellative semigroup that is not a group. Although the semigroup is not assumed to be abelian, we will use + to denote the binary operation. Since the addition is continuous, it is easy to see that given $\{a_n : n \in \omega\}$ and $\{b_n : n \in \omega\}$, $p\text{-}\lim\{a_n + b_n : n \in \omega\} = p\text{-}\lim\{a_n : n \in \omega\} + p\text{-}\lim\{b_n : n \in \omega\}$. S is not a group, therefore, there exist $x \in S$ that does not have inverse, that is, for all $y \in S$ we have $x + y \neq 0$ or $y + x \neq 0$.

Define $f_n \in S^\omega$ as follows:

$$f_n(i) = \begin{cases} (n-i)x & \text{if } n > i \\ 0 & \text{otherwise} \end{cases}.$$

For all $i \in \omega$, let $f(i) = p\text{-}\lim\{f_n(i) : n \in \omega\}$. Note that $f_n(i) = (n-i)x$ for all but finitely many n's. Therefore, $f(i) + ix = p\text{-}\lim\{f_n(i) : n \in \omega\} + ix = p\text{-}\lim\{f_n(i) + ix : n \in \omega\} = p\text{-}\lim\{nx : n \in \omega\} = f(0)$. Hence $(*)$ $\forall i \in \omega f(i) + ix = f(0)$.

From $(*)$, $f(0) = p\text{-}\lim\{f(2i) + (2i)x : i \in \omega\} = p\text{-}\lim\{f(2i) : i \in \omega\} + p\text{-}\lim\{2(ix) : i \in \omega\} = p\text{-}\lim\{f(2i) : i \in \omega\} + p\text{-}\lim\{ix : i \in \omega\} + p\text{-}\lim\{ix : i \in \omega\} = p\text{-}\lim\{f(2i) : i \in \omega\} + 2f(0)$. By the cancelation property, $p\text{-}\lim\{f(2i) : i \in \omega\} + f(0) = 0$.

However, from $(*)$ applied to 1, we have $f(1) + x = f(0)$, therefore, $\big(p\text{-}\lim\{f(2i) : i \in \omega\} + f(1)\big) + x = p\text{-}\lim\{f(2i) : i \in \omega\} + \big(f(1) + x\big) = p\text{-}\lim\{f(2i) : i \in \omega\} + f(0) = 0$. Since S is cancellative, this means that x has inverse, contradiction. ∎

3.2. *Products.* As we mentioned before the proof of Theorem 3.1, no counterexample for the Wallace problem is p-compact.

We recall that if the $2^{\mathfrak{c}}$-th power of a space is countably compact, then the space is p-compact for some free ultrafilter p. Therefore, no counterexample for the Wallace problem can have the $2^{\mathfrak{c}}$-th power countably compact.

From this, it is natural to ask: *which is the minimal power we have to consider in order not to have a counterexample for the Wallace problem?*

As we mentioned earlier we can obtain a counterexample for the Wallace problem under $\mathrm{MA}_{\mathrm{countable}}$ whose square is not countably compact, but it is not known yet whether one can obtain a counterexample for the Wallace problem whose square is countably compact. We have shown in [12] that the ω-th power of a subsemigroup of a free abelian group without non-trivial convergent sequences is not countably compact.

We have also shown that the same is true for semigroups without identity that are subsemigroups of a free abelian group. Therefore, the previously known counterexamples for the Wallace problem, namely the one we have obtained under MA (σ-centered) in [12] and Robbie and Svetlichny's CH example in [10] have the ω-th power not countably compact. Related to this, we also showed in [12] that the ω-th power of every free abelian group is not countably compact.

I believe it is still unknown whether one can find a free abelian group whose square is countably compact. Related to these type of construction, in [13] we have obtained a group under $\mathrm{MA}_{\mathrm{countable}}$ whose square is countably compact but whose cube is not. However that proof relies on the fact that we are working on $2^{\mathfrak{c}}$ so the same arguments would not be enough to get a counterexample for the Wallace problem whose square is countably compact.

PROPOSITION 3.1. *Let H be a topological abelian group and 0 its neutral element. Let $E \ni 0$ be a subsemigroup of H of size at most $\mathfrak{c}$ and G be an ω-bounded subgroup of H. Then $(E + G)^{\mathfrak{c}}$ countably compact implies that $E + G$ is not a counterexample for the Wallace problem.*

PROOF OF THE PROPOSITION. Suppose that $(E+G)^{\mathfrak{c}}$ is countably compact. It suffices to show that $E+G$ is a group, since it is a subgroup of a topological group. Since H is abelian, $E+G$ is a subsemigroup. Clearly $E+G$ is cancellative, since is a subsemigroup of a group. E^ω has size at most $\mathfrak{c}$, since E has size at most $\mathfrak{c}$. For each $n \in \omega$ define $F_n \in (E+G)^{E^\omega}$ as follows:

For each $f \in E^\omega$ let $F_n(f) = f(n)$. By the countable compactness of $(E+G)^{E^\omega}$, there exist $F \in (E+G)^{E^\omega}$ that is an accumulation point of $\{F_n : n \in \omega\}$. Let p be a free ultrafilter such that $F = p\text{-}\lim\{F_n : n \in \omega\}$.

CLAIM 3. *$E+G$ is p-compact.*

PROOF OF THE CLAIM. Let $\{x_n : n \in \omega\}$ be a sequence in $E+G$. Fix $a_n \in E$ and $b_n \in G$ such that $a_n + b_n = x_n$. Let $f \in E^\omega$ such that $f(n) = a_n$, for all $n \in \omega$.

Then $F(f)$ is the $p\text{-}\lim\{F_n(f) : n \in \omega\}$ = p-limit of $\{a_n : n \in \omega\}$. Since G is ω-bounded, there exist $b \in G$ such that b is the p-limit of $\{b_n : n \in \omega\}$. Hence $F(f) + b$ is the the p-limit of $\{x_n : n \in \omega\}$. Since the sequence of x_n's was chosen arbitrarily, we are done. ∎

Now, $E+G$ satisfies the condition from Theorem 3.1, hence $E+G$ is a group and we are done. ∎

An immediate corollary of this:

COROLLARY 3.1. *The Example 1.1 has the $\mathfrak{c}$-th power not countably compact.*

3.3. *Limitations.* We recall that our counterexample for the Wallace problem presented in this paper is, algebraically, a direct sum of $\{ny : n \in \omega\}$ for some $y \in T^{\mathfrak{c}}$ and an ω-bounded group which could be $\{x \in T^{\mathfrak{c}} : \operatorname{supp} x \text{ is bounded}\}$.

In this sense, such a counterexample cannot be achieved in ZFC. Before we give more details, we recall that Kunen's Axiom (KA) is the following statement:

There exist a free ultrafilter p on ω that is generated by a basis of size $\aleph_1$.

Note that CH implies KA trivially. We will use the fact that KA $+\mathfrak{c} = \omega_2$ is consistent (see Exercise VIIIA.10 in [6]).

DEFINITION 3.2. A space is initially ω_1-compact if every open cover of size at most $\aleph_1$ has a finite subcover.

We recall that a space is initially ω_1-compact if and only if every infinite subset X of size at most $\aleph_1$ has a complete accumulation point x, that is for each open neighbourhood U of x, $|X| = |U \cap X|$.

PROPOSITION 3.2 (KA). *Every initially ω_1-compact both-sided cancellative semigroup is a group.*

PROOF OF THE PROPOSITION. If p is any free ultrafilter generated by $\aleph_1$ many elements, then every initially ω_1-compact space is p-compact. We are done by applying Theorem 3.1. ∎

We are ready now to show that the type of counterexample for the Wallace problem we obtained needs some set-theoretic assumption.

THEOREM 3.2 (KA). *Let H be a topological abelian group. Let E be a countable subset of H and G be an ω_1-bounded subgroup of H. Let $S = S_H(E, G)$ be the smallest semigroup containing $E \cup G$. Then S is not a counterexample for the Wallace problem.*

PROOF OF THE THEOREM. Suppose by contradiction that S is a counterexample for the Wallace problem. In particular, S is countably compact. Since E is countable and G is ω_1-bounded, it is easy to show that S is initially ω_1-compact. Then by Proposition 3.2, S is a group, hence a topological group, since it is a subgroup of a topological group, contradiction. ∎

LEMMA 3.3 ($\mathrm{MA}_{\mathrm{countable}} + \mathrm{cf}(\mathfrak{c}) > \aleph_1$). *There exists an initially ω_1-compact counterexample for the Wallace problem.*

PROOF OF THE LEMMA. In Example 2.1, we have seen that under $\mathrm{MA}_{\mathrm{countable}} + \mathrm{cf}(\mathfrak{c}) > \aleph_1$, there exist x and an ω_1-bounded group H (for instance the group generated by the ideal of bounded subsets of $\mathfrak{c}$) such that the semigroup generated was a counterexample for the Wallace problem.

It is not difficult to show that this semigroup is initially ω_1-compact. ∎

THEOREM 3.4. *The existence of an initially ω_1-compact Tychonoff counterexample for the Wallace problem is independent of $\mathfrak{c} = \aleph_2$.*

We could have considered any cardinal arithmetic for $\mathfrak{c}$, as long as KA and $\mathrm{MA}_{\mathrm{countable}}$ are consistent with it and $\mathrm{cf}(\mathfrak{c}) > \aleph_1$.

PROOF OF THE THEOREM. KA and $\mathrm{MA}_{\mathrm{countable}}$ are consistent with $\mathfrak{c} = \aleph_2$. Lemma 3.3 gives a model where such a counterexample exists.

Proposition 3.2 gives us a model where every initially ω_1-compact cancellative semigroup is a group. From results mentioned earlier, such Tychonoff groups are topological groups, hence they are not counterexamples for Wallace's problem. ∎

Related to Theorem 3.4, we have shown in [12] that the existence of a initially ω_1-compact free abelian group is independent of $\mathfrak{c} = \aleph_2$.

COROLLARY 3.2. *Let $G = \{y \in T^{\mathfrak{c}} : \operatorname{supp} y$ is bounded in $\mathfrak{c}\}$.*
Then the following is independent of $\mathfrak{c} = \aleph_2$: There exist $x \in T^{\mathfrak{c}}$ such that the semigroup generated by x and G is a counterexample for the Wallace problem.

FINAL REMARK. D. Grant has also asked in [4] whether a counterexample for the Wallace problem could have the square countably compact or pseudocompact. It is easy to see that our example is pseudocompact for any of its powers, since it contains a dense ω-bounded subgroup.

ACKNOWLEDGEMENTS. I would like also to thank the referee for the helpful suggestions to improve the readability of this paper.

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Canad. Math. Bull. Vol. 39 (4), 1996 pp. 499–506

DEFORMATION OF THE UNIVERSAL ENVELOPING ALGEBRA OF $\Gamma(\sigma_1,\sigma_2,\sigma_3)$

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ABSTRACT. The defining relations for the Lie superalgebra $\Gamma(\sigma_1,\sigma_2,\sigma_3)$ as a contragredient algebra are discussed and a PBW type basis theorem is proved for the corresponding q-deformation.

1. **Introduction.** In this note, we study the q-analog of the universal enveloping algebra of the Lie superalgebra $G = \Gamma(\sigma_1,\sigma_2,\sigma_3)$. This Lie superalgebra is special: as a contragredient algebra, the defining matrix of G over the complex number field C depends on a parameter, the algebra itself already admits a one-parameter deformation. To apply the idea of Drinfeld and Jimbo to define the q-analog of the universal enveloping algebra $U(G)$, one needs to work with a non-integer defining matrix. Hence in general, the deformation is defined over some transcendental function field extension of C (or just the field C, if one takes the deformation parameter to be a suitable complex number). The deformation thus defined will actually be a two-parameter family of algebras.

We discuss the defining relations for G as a contragredient algebra in Section 2. Although these defining relations are known to the experts (*cf.* the discussion in [8]), we are unable to find a suitable reference, so we provide a complete proof for these relations.

In Section 3, we define the deformation $\mathcal{U}$ of $U(G)$ and study its structure. As in the other cases of type II classical contragredient Lie superalgebras (see [4] for the definition of type II Lie superalgebras, see [5] for a definition of the q-deformation of $U\big(\mathrm{osp}(m,2n)\big)$, the usual Drinfeld-Jimbo deformation of $U(G)$ does not contain a copy of the standard deformation of $U(G_0)$, where G_0 is the even part of G, since there are not enough group like elements in it. However, we show that in our case, one can introduce suitable elements in $\mathcal{U}$ such that a PBW type theorem (Theorem 3.3) holds for $\mathcal{U}$.

2. **The defining relations for** G. We use the notation adopted in [9]. Recall that the algebra G is defined as a contragredient Lie superalgebra with three nonzero elements $\sigma_1,\sigma_2,\sigma_3 \in$ C satisfying $\sigma_1+\sigma_2+\sigma_3 = 0$, with generators e_i, f_i, h_i $(i = 1,2,3)$ and the defining matrix $(a_{ij})_{3\times 3}$ given as follows:

$$\begin{pmatrix} 0 & 2\sigma_2 & 2\sigma_3 \\ -1 & 2 & 0 \\ -1 & 0 & 2 \end{pmatrix}.$$

Received by the editors October 2, 1995.

AMS subject classification: 16W30, 16S80, 22E70.

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The grading on G is given by

$$\deg h_i = 0, i = 1,2,3; \quad \deg e_i = \deg f_i = 0, i = 2,3; \quad \deg e_1 = \deg f_1 = 1.$$

PROPOSITION 2.1. *The defining relations for G as a contragredient Lie superalgebra are*

(1) $[h_i, h_j] = 0, \quad i,j = 1,2,3;$
(2) $[h_i, e_j] = a_{ij}e_j, \quad [h_i, f_j] = -a_{ij}f_j, \quad i,j = 1,2,3;$
(3) $[e_i, f_j] = \delta_{ij}h_i, \quad i,j = 1,2,3;$
(4) $(\mathrm{ad}\, e_i)^{1-a_{ij}}(e_j) = 0, \quad (\mathrm{ad} f_i)^{1-a_{ij}}(f_j) = 0, \quad i = 2,3, \quad j = 1,2,3;$
(5) $[e_1, e_1] = 0, \quad [f_1, f_1] = 0.$

Relations (1)–(5) clearly hold in $\Gamma(\sigma_1, \sigma_2, \sigma_3)$, so we assume that G is defined as a contragredient Lie superalgebra by using the given generators and these relations and show that G is isomorphic to $\Gamma(\sigma_1, \sigma_2, \sigma_3)$. The proof will be organized in several lemmas.

Note that by the Jacobi identity, we have

$$\big[e_1, [e_1, e_i]\big] = 0, \quad \big[f_1, [f_1, f_i]\big] = 0, \quad i = 2,3.$$

Let

$$\begin{aligned} e_0 &= (2\sigma_1)^{-1}\Big[e_1, \big[e_3, [e_2, e_1]\big]\Big] \\ &= (2\sigma_1)^{-1}\big[[e_1, e_3], [e_2, e_1]\big], \\ f_0 &= (2\sigma_1)^{-1}\Big[f_1, \big[f_3, [f_2, f_1]\big]\Big] \\ &= (2\sigma_1)^{-1}\big[[f_1, f_3], [f_2, f_1]\big], \\ h_0 &= [e_0, f_0]. \end{aligned}$$

LEMMA 2.2. *The subalgebra $\langle e_0, f_0, h_0\rangle$ of G generated by e_0, f_0, h_0 is isomorphic to* sl(2), *and* $\langle e_i, f_i, h_i;\ i = 0,2,3\rangle \cong \mathrm{sl}(2) \oplus \mathrm{sl}(2) \oplus \mathrm{sl}(2)$.

PROOF. A straightforward computation shows that

$$h_0 = (2\sigma_1)^{-1}(2\sigma_2 h_2 + 2\sigma_3 h_3 - 2h_1).$$

Hence $[h_0, e_0] = 2e_0$, $[h_0, f_0] = -2f_0$, and $\langle e_0, f_0, h_0\rangle \cong \mathrm{sl}(2)$. For the second statement, first we note that by the definitions of e_0 and f_0, we have

$$[e_0, f_i] = 0, \quad [f_0, e_i] = 0, \quad i = 2,3.$$

Then we note that

$$\begin{aligned}[e_2,e_0] &= (2\sigma_1)^{-1}\Big[e_2,\big[[e_1,e_3],[e_2,e_1]\big]\Big]\\ &= (2\sigma_1)^{-1}\Big[\big[e_2,[e_1,e_3]\big],[e_2,e_1]\Big]\\ &= -(2\sigma_1)^{-1}\Big[\big[e_3,[e_2,e_1]\big],[e_2,e_1]\Big]\\ &= -(4\sigma_1)^{-1}\Big[e_3,\big[[e_2,e_1],[e_2,e_1]\big]\Big]\\ &= -(4\sigma_1)^{-1}\Big[e_3,\Big[e_2,\big[e_1,[e_2,e_1]\big]\Big]\Big]\\ &= 0,\end{aligned}$$

and similarly

$$[e_3,e_0]=0,\quad [f_2,f_0]=0,\quad [f_3,f_0]=0.$$

Now the lemma follows from these identities.

Let $G_0 = \langle e_i,f_i,h_i;\ i=0,2,3\rangle$ (Lemma 2.4 below will show that G_0 is indeed the even part of G and thus justify our notation).

LEMMA 2.3. *Let* $e_{111}=\big[e_3,[e_2,e_1]\big]$, *then as a* G_0*-module via the adjoint representation, the submodule* (e_{111}) *generated by* e_{111} *is isomorphic to* $\mathbb{C}^2\otimes\mathbb{C}^2\otimes\mathbb{C}^2$, *where* $\mathbb{C}^2$ *is the two-dimensional natural representation of* sl(2).

PROOF. By the definition of e_{111}, $e_{111}\neq 0$. Note that we have $[e_i,e_{111}]=0, i=2,3$. Note also that $[e_0,e_1]=0$, so since $[e_i,e_j]=0$ for $i,j\neq 1$, we see that

$$\begin{aligned}[e_0,e_{111}] &= \Big[e_0,\big[e_3,[e_2,e_1]\big]\Big]\\ &= \Big[e_3,\big[e_2,[e_0,e_1]\big]\Big]\\ &= 0.\end{aligned}$$

Thus e_{111} is a highest weight vector. Now since $[h_i,e_{111}]=e_{111}$ and $(\mathrm{ad} f_i)^2(e_{111})=0$ $(i=0,2,3)$, by the representation theory of the semisimple Lie algebras, the lemma follows as desired.

Define the following elements of G:

$$e_{112}=[f_3,e_{111}],\quad e_{121}=[f_2,e_{111}],$$

$$f_{212}=[f_3,f_1],\quad f_{221}=[f_2,f_1],\quad f_{222}=\big[f_3,[f_2,f_1]\big].$$

Then e_1,f_1, together with the e_{ijk} and the f_{rst} form a basis of the G_0-module (e_{111}).

LEMMA 2.4. *Table I in [9] holds for the elements we defined above, where* (ijk) *corresponds to the* e_{ijk} *or the* f_{ijk} *with* $(122)\leftrightarrow e_1$ *and* $(211)\leftrightarrow f_1$.

PROOF. We only verify that $e_{111}^2=0$, the other relations can be verified similarly. Since $[e_1,e_1]=0$, we have

$$\big[e_2,[e_1,e_1]\big]=2\big[[e_2,e_1],e_1\big]=0,$$

and hence by applying ad e_2 and using (4) in Proposition 2.1, we have

$$\big[[e_2,e_1],[e_2,e_1]\big]=0.$$

Therefore

$$x=:\Big[\big[e_3,[e_2,e_1]\big],[e_2,e_1]\Big]=1/2\Big[e_3,\big[[e_2,e_1],[e_2,e_1]\big]\Big]=0,$$

and thus

$$[e_{111},e_{111}]=[e_3,x]=0.$$

PROOF OF PROPOSITION 2.1. The proposition follows from Lemma 2.2-Lemma 2.4 with G_0 being the even part of $\Gamma(\sigma_1,\sigma_2,\sigma_3)$, (e_{111}) being the odd part of $\Gamma(\sigma_1,\sigma_2,\sigma_3)$ (for the structure of $\Gamma(\sigma_1,\sigma_2,\sigma_3)$, see [9, Section 2]).

3. **Deformation of $U(G)$ and a PBW type theorem.** Let q be a variable over $\mathbb{C}$, and let $q_1=q^{-1}$, $q_i=q^{2\sigma_i}$ $(i=2,3)$ (the q_i are well defined complex value functions as long as $q\neq 0$). Let $\mathcal{A}=\mathbb{C}[q^{\pm1};q_i^{\pm1},i=2,3]$, and let $\mathcal{F}$ be the quotient field of $\mathcal{A}$.

We define the algebra $\mathcal{U}$ to be the $\mathbb{Z}_2$-graded associative algebra with 1 over $\mathcal{F}$ generated by the elements E_i, F_i, $K_i^{\pm1}$ $(i=1,2,3)$, with grading given by

$$\begin{gathered}\deg E_i=\deg F_i=0,\quad i=2,3;\quad \deg K_i^{\pm1}=0,\quad i=1,2,3;\\ \deg E_1=\deg F_1=1,\end{gathered}$$

and with the following generating relations:

$$K_iK_j=K_jK_i,\quad K_iK_i^{-1}=K_i^{-1}K_i=1,\quad 1\le i,j\le 3, \tag{3.1}$$

$$K_iE_jK_i^{-1}=q_i^{a_{ij}}E_j,\quad K_iF_jK_i^{-1}=q_i^{-a_{ij}}F_j,\quad 1\le i,j\le 3, \tag{3.2}$$

$$\begin{gathered}E_iF_j-(-1)^{ab}F_jE_i=\delta_{ij}\frac{K_i-K_i^{-1}}{q_i-q_i^{-1}},\\ a=\deg E_i,\quad b=\deg F_j,\quad 1\le i,j\le 3.\end{gathered} \tag{3.3}$$

$$E_2E_3=E_3E_2,\quad F_2F_3=F_3F_2, \tag{3.4}$$

$$\begin{gathered}E_i^2E_1-(q_i+q_i^{-1})E_iE_1E_i+E_1E_i^2=0,\quad i=2,3,\\ F_i^2F_1-(q_i+q_i^{-1})F_iF_1F_i+F_1F_i^2=0,\quad i=2,3,\end{gathered} \tag{3.5}$$

$$E_1^2=F_1^2=0. \tag{3.6}$$

The algebra $\mathcal{U}$ is a $\mathbb{Z}_2$-graded Hopf algebra with comultiplication Δ, antipode S and counit ε defined by

$$\Delta E_i = E_i \otimes 1 + K_i \otimes E_i, \quad \Delta F_i = F_i \otimes K_i^{-1} + 1 \otimes F_i, \quad \Delta K_i = K_i \otimes K_i; \tag{3.7}$$

$$SE_i = -K_i^{-1} E_i, \quad SF_i = -F_i K_i, \quad SK_i = K_i^{-1}; \tag{3.8}$$

$$\varepsilon E_i = 0, \quad \varepsilon F_i = 0, \quad \varepsilon K_i = 1. \tag{3.9}$$

There exists a C-algebra anti-automorphism θ of $\mathcal{U}$ given by

$$\theta E_i = F_i, \quad \theta F_i = E_i, \quad \theta K_i = K_i^{-1}, \quad \theta q = q^{-1} \tag{3.10}$$

and $\theta(uv) = \theta(v)\theta(u)$, for all $u, v \in \mathcal{U}$.

The adjoint action of $\mathcal{U}$ on itself is given by

$$\mathrm{ad}_q x(y) = \sum (-1)^{\deg(b_i)\deg(y)} a_i y S(b_i), \tag{3.11}$$

where $\Delta x = \sum a_i \otimes b_i$. Note that by using the adjoint action, relations (3.4) and (3.5) can be replaced by

$$(\mathrm{ad}_q E_i)^{1-a_{ij}} E_j = 0, \quad i = 2, 3, \quad 1 \le j \le 3. \tag{3.12}$$

Introduce the following elements of $\mathcal{U}$:

$$\begin{aligned} E_{121} &= \mathrm{ad}_q E_3(E_1) = E_3 E_1 - q_3^{-1} E_1 E_3, \\ E_{112} &= \mathrm{ad}_q E_2(E_1) = E_2 E_1 - q_2^{-1} E_1 E_2, \\ E_{111} &= \mathrm{ad}_q E_3 \mathrm{ad}_q E_2(E_1) = \mathrm{ad}_q(E_3 E_2)(E_1), \\ E_0 &= (q_2 + q_2^{-1}) E_1 E_{111} + (q_3 + q_3^{-1}) E_{111} E_1 + (q_3 q_2^{-1} - q_3^{-1} q_2) E_{121} E_{112}, \end{aligned} \tag{3.13}$$

and let

$$F_{212} = \theta E_{121}, \quad F_{221} = \theta E_{112}, \quad F_{222} = \theta E_{111}, \quad F_0 = \theta E_0. \tag{3.14}$$

LEMMA 3.1. *The following formulas hold in $\mathcal{U}$:*

(1) $E_{ijk}^2 = 0$, $F_{ijk}^2 = 0$,
(2) $E_1 E_{121} + q_3 E_{121} E_1 = 0$, $F_1 F_{212} + q_3 F_{212} F_1 = 0$,
(3) $E_1 E_{112} + q_2 E_{112} E_1 = 0$, $F_1 F_{221} + q_2 F_{221} F_1 = 0$,
(4) $E_{112} E_{111} + q_3 E_{111} E_{112} = 0$, $F_{221} F_{222} + q_3 F_{222} F_{221} = 0$,
(5) $E_{121} E_{111} + q_2 E_{111} E_{121} = 0$, $F_{212} F_{222} + q_2 F_{222} F_{212} = 0$,
(6) $E_1 E_{111} + q^{-2\sigma_1} E_{111} E_1 + q_2 E_{112} E_{121} + q_3 E_{121} E_{112} = 0$.

PROOF. We only need to prove those formulas involving E, those involving F can then be obtained by applying θ. Formulas (2) and (3) are clear, (4) and (5) can be verified by using $E_{111} = \mathrm{ad}_q E_2(E_{121})$ or $E_{111} = \mathrm{ad}_q E_3(E_{112})$, (6) can be verified by using (2) and (3). To verify (1), note that formulas (3.5) and (3.6) imply that

$$E_1E_i^2E_1 = (q_i + q_i^{-1})E_1E_iE_1E_i = (q_i + q_i^{-1})E_iE_1E_iE_1, \quad i = 2,3.$$

Thus $E_{121}^2 = 0$, $E_{112}^2 = 0$. Similarly, using $E_{112}^2 = 0$ and $(\mathrm{ad}_q E_3)^2 E_{112} = 0$ instead of (3.5) and (3.6), we get

$$E_{111}^2 = \left(\mathrm{ad}_q E_3(E_{112})\right)^2 = 0.$$

The proof of the lemma is now complete.

The following lemma provides some formulas involving the element E_0.

LEMMA 3.2. *The following formulas hold in* $\mathcal{U}$:

(1) $E_0E_2 = E_2E_0$, $E_0E_3 = E_3E_0$, $E_0F_2 = F_2E_0$, $E_0F_3 = F_3E_0$,
(2) $E_0E_1 - q^{2\sigma_1}E_1E_0 = q_2(1 - q^{4\sigma_1})E_1E_{111}E_1$,
(3) $E_0E_{121} = E_{121}E_0$, $E_{112}E_0 = E_0E_{112}$, $E_0E_{111} = E_{111}E_0$.

PROOF. The proofs for those formulas involving only the E's are just direct applications of Lemma 3.1. To verify the last two formulas in (1), we use the following formulas

$$\begin{aligned} F_2E_{112} - E_{112}F_2 &= E_1K_2^{-1}, \\ F_2E_{121} &= E_{121}F_2, \\ F_2E_{111} - E_{111}F_2 &= E_{121}K_2^{-1}, \end{aligned} \tag{3.15}$$

$$\begin{aligned} F_3E_{112} &= E_{112}F_3, \\ F_3E_{121} - E_{121}F_3 &= E_1K_3^{-1}, \\ F_3E_{111} - E_{111}F_3 &= E_{112}K_3^{-1}. \end{aligned} \tag{3.16}$$

REMARK. Compare with the corresponding formulas in $U(G)$, one would like to have a vector E_0 which satisfies (1) in Lemma 3.2 and has a better commutation relation with E_1, but this does not seem to be possible, since a search along this line will lead to the left hand side of (6) in Lemma 3.1, which is 0.

Let $\mathcal{U}_{\mathcal{A}}$ be the $\mathcal{A}$-subalgebra of $\mathcal{U}$ generated by E_i, F_i, $K_i^{\pm 1}$ and

$$[K_i; 0] = \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}, \quad i = 1,2,3.$$

For $\epsilon \in \mathbb{C}^\times$, let $\mathcal{U}_\epsilon = \mathcal{U}_{\mathcal{A}}/(q - \epsilon)\mathcal{U}_{\mathcal{A}}$. Then the algebra $\mathcal{U}_1$ is an associative algebra over $\mathbb{C}$ with generators E_i, F_i, K_i, $H_i = [K_i; 0]$ $(i = 1,2,3)$ and the defining relations (which can be verified easily):

$$K_i \text{ are central elements with } K_i^2 = 1, \tag{3.17}$$

$$[E_i,F_j]=\delta_{ij}H_i,\quad [H_i,E_j]=a_{ij}K_iE_j,\quad [H_i,F_j]=-a_{ij}K_iF_j, \tag{3.18}$$

$$(\operatorname{ad}E_i)^{1-a_{ij}}(E_j)=0,\quad (\operatorname{ad}F_i)^{1-a_{ij}}(F_j)=0,\quad i=2,3,\quad j=1,2,3, \tag{3.19}$$

$$E_1^2=0,\quad F_1^2=0. \tag{3.20}$$

Therefore, $\mathcal{U}_1/(K_i-1;\,i=1,2,3)\cong U(G)$, the universal enveloping algebra of G. Note that the image of E_0 in $U(G)$ is $2e_0$, where e_0 is defined in Section 2.

Let $\mathcal{U}^+$, $\mathcal{U}^-$, $\mathcal{U}^0$ be the subalgebras of $\mathcal{U}$ generated by the E_i, the F_i, and the $K_i^{\pm1}$ $(i=1,2,3)$ respectively. Then just as in the Lie algebra case (see [7]), one can show that $\mathcal{U}=\mathcal{U}^-\mathcal{U}^0\mathcal{U}^+$ and (use the comultiplication) that $\mathcal{U}\cong\mathcal{U}^-\otimes\mathcal{U}^0\otimes\mathcal{U}^+$ as $\mathcal{F}$-vector spaces.

For $\delta=(\delta_1,\delta_2,\delta_3,\delta_4)$, $\delta_i=0$ or 1; $m=(m_1,m_2,m_3), m_i\in\mathbb{Z}_+$, let

$$\begin{aligned}E^{(\delta,m)}&=E_{111}^{\delta_1}E_{121}^{\delta_2}E_{112}^{\delta_3}E_1^{\delta_4}E_0^{m_1}E_2^{m_2}E_3^{m_3},\\ F^{(\delta,m)}&=F_{222}^{\delta_1}F_{212}^{\delta_2}F_{221}^{\delta_3}F_1^{\delta_4}F_0^{m_1}F_2^{m_2}F_3^{m_3}.\end{aligned} \tag{3.21}$$

For $t=(t_1,t_2,t_3)$, $t_i\in\mathbb{Z}$, let

$$K^t=K_1^{t_1}K_2^{t_2}K_3^{t_3}. \tag{3.22}$$

Then the K^t form a basis of $\mathcal{U}^0$, and we have the following theorem:

THEOREM 3.3. *The elements of the form $E^{(\delta,m)}$ (resp. $F^{(\delta,m)}$) form a basis of $\mathcal{U}^+$ (resp. $\mathcal{U}^-$), and the elements of the form*

$$F^{(\delta,m)}K^tE^{(\delta',m')}$$

form a basis of $\mathcal{U}$.

PROOF. We only need to prove that the elements of the form $E^{(\delta,m)}$ form a basis of $\mathcal{U}^+$, since the statement about $\mathcal{U}^-$ will follow from symmetry and the statement about $\mathcal{U}$ will follow from the fact that $\mathcal{U}\cong\mathcal{U}^-\otimes\mathcal{U}^0\otimes\mathcal{U}^+$. We first show that these elements span $\mathcal{U}^+$, that is, by using the commutation relations in $\mathcal{U}^+$ we can express any monomial of $\mathcal{U}^+$ as a linear combination of these elements. In fact, Lemma 3.1 and Lemma 3.2 along with the defining relations of $\mathcal{U}$ provide the commutation relations we need. In particular, to bring the terms $E_{112}E_{121}$ and E_1E_{111} to the right order, we use formula (6) in Lemma 3.1 together with the definition of E_0. Then, we show that these elements are linearly independent over $\mathcal{F}$. Note that by (3.13), these elements are in fact in $\mathcal{U}_{\mathcal{A}}$. So if we have a linear relation

$$\sum_{i=1}^{r}c_iE^{(\delta_i,m_i)}=0, \tag{3.23}$$

with $0 \neq c_i \in \mathcal{F}$ $(1 \leq i \leq r)$, then by multiplying a suitable element from $\mathcal{A}$, we may assume that $c_i \in \mathcal{A}$. Now if there exists a c_i such that $c_i(1) \neq 0$, then the image of the right hand side of (3.20) gives a nontrivial linear relation in $U(G)$. But by the PBW theorem of $U(G)$, the images of the $E^{(\delta,m)}$ in $U(G)$ form a basis of $U(G)$, and we have a contradiction. If $c_i(1) = 0$ for all $1 \leq i \leq r$, then by the results in [1, Ch. 3, Section 3], we may assume that the order of 1 for c_i is n_i, and set $n = \min\{n_i : 1 \leq i \leq r\}$. Then $\lim_{q\to 1} c_i/(q-1)^n \neq 0$ for some i, hence by (3.20) we have

$$\lim_{q\to 1}\left(\frac{1}{(q-1)^n}\sum_{i=1}^{r} c_i E^{(\delta_i,m_i)}\right) = \sum_{i=1}^{r}\left(\lim_{q\to 1}\frac{c_i}{(q-1)^n}\right)E^{(\delta_i,m_i)} = 0,$$

which provides a nontrivial linear relation in $U(G)$ contradicting the PBW theorem for $U(G)$. Hence the elements of the form $E^{(\delta,m)}$ are linearly independent, and the proof of the theorem is now complete.

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Publié par la Société mathématique du Canada

Canadian Mathematical Bulletin

canadien de mathématiques

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POUR LA SOCIÉTÉ MATHÉMATIQUE DU CANADA
IMPRIMÉ AU CANADA PAR LES PRESSES DE L'UNIVERSITÉ DE TORONTO

CN ISSN 0008-4395

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